Triangulated ladders, polynomials and links.

by

ALFRED NTAZINDA

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Declaration.

I, Alfred Ntazinda declare that this is my own work and the references has been made to wherever other people's work has been cited. I also declare that this work has not been presented or submitted for any award elsewhere.

Alfred Ntazinda: ....

Date:

....

Certificate.

This is to certify that the work in the thesis entitled, "Triangulated Ladders, Polynomials and Links" has been written under my supervision, and it has not been published in any form elsewhere.

Dr. Eunice Mphako-Banda..
The Supervisor.
Date:

Dedication.

To my children: Parfait, Joelle and Romeo, and to my wife Agnes.

Abstract.

In this thesis we study three graph polynomials, namely, the chromatic polynomial, the Tutte polynomial and the coboundary polynomial. We define a class of graphs which we call triangulated ladder and give some of its properties. Then we find an explicit expression of the chromatic polynomial for this class of graphs. Furthermore, we find a recursive expression of the Tutte polynomial and a recursive expression of the coboundary polynomial for this class of graphs. Finally we give a class of links associated with triangulated ladders and give some of their properties.

Acknowledgments.

I wish to thank my supervisor EUNICE MPHAKO-BANDA who helped me in the preparation of this thesis. I would like to express my sincere gratitude to the government of Rwanda which gave me a scholarship. I wish also to thank Chancellor College in general for its hospitality especially the staff and students in the Department of Mathematical Sciences for their warmness. Many thanks also to the KIST Department of Mathematics for their assistance in putting at my disposal all resources for preparing my thesis.

Contents

CHAPTER 1

Introduction.

In this thesis we target to compute polynomials whose importance is crucial in combinatorial theory, namely in graph theory, and in knot theory. In graph theory, it is usual to associate invariants to combinatorial objects, in particular to graphs, in order to study some of their properties. These invariants can be of various types: structural, numerical, algebraic, or polynomial. The Tutte polynomial is a two variables polynomial originally defined for graphs by Tutte and Whitney and later generalized to matroids by Crapo (1969). It was first conceived as an extension of the chromatic polynomial, but nowadays it is known to have applications in many areas of combinatorics and other areas of mathematics. One of its striking features is that it contains a great deal of information about the underlying graph. For instance, from the knowledge of the Tutte polynomial, one can deduce enumerative results on bases, colourings, and orientations, and also structural properties such as connectivity .

The origins of a mathematical theory of knots can be traced back to the German mathematician Carl Friedrich Gaub , who tried to classify closed plane curves with a finite number of self-intersections, which he sometimes called "Tractfiguren" see de Mier (2003). However, we do know that Gaub worked on his Tractfiguren in 1825 and 1844. While Gaub may not have taken a long-term active interest in studying what we now recognize as knots, his student Johann Benedikt Listing did.

Listing's 1847 "Vorstudien zur Topologie", in which he first coined the term "topology," included a discussion of mathematical knots and their classification. Listing was interested in developing an algebraic calculus of knot diagrams so that it could easily be determined when two diagrams represented the same knot. However, the way he couched the problem prevented him from proving any useful results in knot theory. The first work on knot theory outside of Germany began in Scotland in the late 1860s as the physicist William Thomson (later Lord Kelvin) began looking for a suitable atomic theory. In 1867, Thomson, who was inspired by Hermann von Helmholtz's work on vortex motion and a demonstration by Peter Guthrie Tait exhibiting the properties of vortices using smoke rings, presented a paper to the Royal Society of Edinburgh proposing that atoms were knotted vortices. Thomson continued thinking about atoms as vortices, sparking the interest of James Clerk Maxwell. In the fall of 1868, Maxwell began to undertake a serious study of topology see Maxwell (1995,1862-1873). Specifically, Maxwell wanted to know when two projections of a link represented the same link in 3-dimensional space. He did observations that regions bounded by three or fewer arcs were sufficient to transform any projection of a link into any other projection. However, nearly sixty years later the German mathematician Kurt Reidemeister would prove this very fact, and today the diagrammatic "moves" discovered by Maxwell bear Reidemeister's name. P.G. Tait by 1876 had set out to make a complete table of knots (up to a certain number of crossings). The one major positive result that came out of Tait's initial work on knot enumeration was the establishment of the existence of knots that could be deformed from right-handed to left-handed without changing the structure of the knot. He called these knots amphicheiral see Murasugi (1987), a term that survives to this day (coexisting with the term achiral), and recognized that the figure-8 was amphicheiral. The idea of chirality is important in modern applications of knot theory.

In the Knot Theory until 1984 the main tool to tell the knots apart was the Alexander polynomials so named after the American mathematician J.W.Alexander. However, those did not distinguish between the two trefoil knots. For both knots the Alexander polynomial is the same.

In 1984 a New Zealander Vaughan Jones working on some aspects of Mathematical Physics discovered (Jones) polynomials that later were generalized even further simultaneously and quite independently by five separate groups of mathematicians. Known as the HOMFLY (Hoste-Ocneanu-Millett-Freyd-Lickorish-Yetter) see Hoste (to appear), these polynomials in two variables give different equations of the left and the right trefoil knots, respectively.

Nowadays, relationship between knot and graph polynomials is established, for example, it is well-known that the Jones polynomial of an alternating knot is closely related to the Tutte polynomial of a special graph obtained from a regular projection of the knot and a celebrated result of F. Jaeger states that the Tutte polynomial of a planar graph is determined by the HOMFLY polynomial of an associated link see Jaeger (1988,no.2, 647-654). Now, concerning the "Triangulated graphs", it is known that decomposable models are a subset of undirected graphical models that are built from triangulated graphs. A graph is triangulated (chordal, decomposable) if every cycle of length four or greater contains a chord. Models of this type posses a number of desirable qualities see Deshpande and Garofarakis and Jordan (2001), including:

(1) Maximum likelihood estimates can be calculated directly from marginal probabilities, eliminating the need for Iterative Proportional Fitting procedures,
(2) Closed form expressions for test statistics can be found,
(3) Every decomposable model can be represented as either a directed or undirected model,
(4) Inference algorithms for decomposable graphs are tractable.

Many Mathematicians are writing on triangulated graphs: Jayson Rome wrote on "Graph triangulation" in order to decompose models built from triangulated graphs. F. R. K. Chung and David Mumford wrote on "Chordal completions of planar graphs" to finding for a given graph, a chordal completion with as few edges as possible. They were motivated by applications in computer vision and artificial intelligence. Broderick Arneson and Piotr Rudnicki worked to "Recognizing Chordal Graphs: Lex BFS and MCS1" for formalizing the algorithm for recognizing chordal graphs... Many authors are interested in triangulated graphs that myself I was interested in a particular class of triangulated graph, a triangulated ladder. The importance of graph polynomials is now well known in terms of information of properties they encode, and relationship between different polynomials is established, one can derive a diagram knot from an appropriate graph see Kauffman and Murasugi (1989) and this process is reversible see Noble and Welsh (1999) and Traldi (1989). The challenge is: given any graph, are we able to produce its graph polynomials? This thesis is not to supply the key issue to that problem, but our simple contribution is limited only to compute some of the main polynomials to a class of graph we called Triangulated ladders, since these polynomials encode a good number of properties for the underlying graph; and then, the knot associated.

The most powerful tool we are using in this thesis, is to define a tree diagram from the formula of deletion contraction of chromatic, coboundary and Tutte polynomials and then, to gather the results in suitable tableaux so that we determine the graph polynomials at ease.

CHAPTER 2

Basic notions and definitions.

In this chapter, first we give an overview of the thesis and then after, we give concepts and basic definitions we will need in this work.

2.1. Overview of thesis.

In Chapter 1, we give an introduction to the thesis. In Chapter 2, we give basic notions and definitions that are relevant to this thesis. In Section 2.2, we give concepts, basic notions, definitions and notation that we will need in this thesis. These are: notions of graph theory, matroid notions, deletions and contractions in graphs, graph polynomials, the definition of a triangulated ladder. We close this section by giving examples and notation for some triangulated ladders of parallel classes. In Chapter 3, we give the edge set, the chromatic and the coboundary polynomials of a triangulated ladder. In Section 3.1, we calculate the size of the edge set of a triangulated ladder, in Section 3.2, we compute the expression of the Chromatic polynomial of a triangulated ladder in both forms, recursive and explicit and in Section 3.3, we compute the expression of the Coboundary polynomial of a triangulated ladder using a tree diagram, then we give some examples and we exhibit the relationship between the coboundary and the chromatic polynomials. In Chapter 4, we give an expression of the Tutte polynomial and discuss on a link associated to a triangulated ladder: in Section 4.1, we compute the Tutte polynomial for a triangulated n-ladder using a tree diagram, then we give some examples and in Section 4.2, we give the links associated with a triangulated ladder: first we give a general definition of links, then we construct a link from a planar graph and finally, we give the component number of a link associated with a triangulated ladder. Finally in Chapter 5, we give the conclusion of the thesis.

2.2. Concepts, basic definitions.

In this section, we give notions of graph theory, especially planar graphs that are relevant to this work. Then, we give basic notions of matroids, in particular we focus on the relationship between graphs and matroids that are relevant to this thesis. We recall the notions of deletion and contraction in a graphs and we are giving also definitions and theorems stating the main polynomials that we will compute. In this section we give also the definition of a triangulated ladder, and we close it with the notation to be used throughout the thesis.

2.2.1. Notions of graph theory. The notions defined in this subsection are widely known in graph theory; we refer the reader to [**?**] for further details. A *graph* is a triple consisting of a vertex set $V(G)$, an *edge set* $E(G)$ and a relation that associates with each *edge* two vertices called its *endpoints*. If two vertices u and v are endpoints of an edge, they are *adjacent* and are *neighbors*. If a vertex v is an endpoint of an edge e then v and e are *incident*. A *subgraph* H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We then, write $H \subseteq G$ and say that G *contains* H. A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same endpoints. Multiple edges are also known as *parallel edges*. A *simple graph* is a *graph* having no loops or multiple edges. A *path*, as defined by West see Douglas (1996), is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A *walk* is a list $v_0, e_1, v_1, e_2, v_2, \ldots, e_i, v_i, \ldots, v_k, e_k$, for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1}, v_i. The *length* of a walk (a path) is its number of edges. A u, v-*walk* has the first vertex u and the last vertex v. These are its endpoints. A walk is *closed* if its endpoints are the same. The *degree* of a vertex is the number of incident edges. A *cycle* of a graph is a closed walk all of whose vertices have degree 2. A graph is *connected*, if it has an $u, v-$path whenever $u, v \in V(G)$. A graph with no cycle is *acyclic*. A *forest* is an acyclic graph. A *tree* is a connected acyclic graph. A *leaf* is a vertex of degree one. A *spanning subgraph* is a subgraph with vertex set V(G). A *spanning tree* is a spanning subgraph that is a spanning tree. For the following theorem we refer the reader to Douglas (1996) for more details.

Theorem 2.2.1. *An n vertex graph G for* $n \geq 1$, *is a tree if and only if G is connected and has* $n - 1$ *edges* .

A *curve* is the image of a continuous map from $[0, 1]$ *to* $\mathbb{R}^n$. A *polygonal curve* is a curve composed of the finitely many line segments. A *polygonal is* u, v-*curve* when it starts at u and ends at v. A *drawing* of a graph G is a function f defined on $V(G) \cup E(G)$ that assigns each vertex v, a point $f(v)$ in the plane and assigns each edge with endpoints u, v a polygonal $f(u), f(v)$ -curve. The images of vertices are distinct. A point in $f(e) \cap f(e')$ that is not a common endpoint is a *crossing*. A graph is a *planar* if it has a drawing without crossings such a drawing is a *planar embedding* of G. A *planar graph* is a particular embedding of a planar graph. A *curve is closed* if its first and last point are the same. It is *simple* if it has no repeated points except possibly the first and the last. An *open set* in the plan is a set $U \subseteq \mathbb{R}^n$ such that for every $p \in U$, all within small distance from p belong to U. A *region* is an open set U that contains a polygonal u, v -curve for every pair $u, v \in U$.

2.2.2. Matroid notions. The basic matroid notions defined in this subsection can be found in Oxley (1992). A *matroid* is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E satisfying the following conditions.

(1) $\emptyset \in \mathcal{I}$;
(2) If $I \in \mathcal{I}$ and $I' \subseteq I$ then $I' \in \mathcal{I}$;
(3) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$ then, there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

If M is a matroid $(E, \mathcal{I})$, then M is called a *matroid on* E. The members of $\mathcal{I}$ are called the *independent sets* of M. A member of E, that is not in $\mathcal{I}$ is called *dependent*. A maximal dependent set in a matroid is called *a circuit of a matroid*. We denote the set of circuits of M by $\mathscr{C}$ or $\mathscr{C}(M)$. The members of $\mathcal{I}(M)$ are those subsets of $E(M)$ that contain no member of $\mathscr{C}(M)$. A maximum independent set in M is called a *basis of M*. We start with a graph and we define a matroid by means of a theorem, whose set of circuits is the set of cycles of the edge set of the graph. Then we define the rank of a matroid thus the rank of the corresponding graph. The following theorem is well known in the literature; for more details see Oxley (1992).

THEOREM 2.2.2. *Let E be the set of edges of a graph G and $\mathscr{C}$ be the set of edges of sets of cycles of G, then $\mathscr{C}$ is the set of circuits of a matroid on E.*

A Matroid $M(G)$ derived above from the graph is called a *cycle matroid*. A set X of edges is independent in $M(G)$ if and only if X does not contain the edge set of a cycle, or equivalently, $G[X]$, the subgraph induced by X is a forest. Let M be a matroid $(E, \mathcal{I})$, suppose that $X \subseteq E$. Let $\mathcal{I}|X$ be $\{I \subseteq X : I \in \mathcal{I}\}$. Then it is easy to see that $(X, \mathcal{I}|X)$ is a matroid . We call this matroid the restriction of M to X, denote $M|X$. One can check easily that $\mathscr{C}(M|X) = \{C \subseteq X : C \in \mathscr{C}(M)\}$. We define the *rank* $r(X)$ of X to be the size of a basis B of $M|X$. The function r maps 2^E into the set of nonnegative integers. This function is called the *rank function* of M, we shall usually denote $r(M)$ for $r(E(M))$ and sometimes when there is no confusion we denote it r.

2.2.3. Deletions and contractions in graphs. In this thesis, we often use the operations of deletion and contraction of edges in a graph G. These operations are widely known in graph theory, for example see Douglas (1996). In a graph G, contracting an edge e with endpoints u and v is the replacement of u and v with a single vertex whose incident edges are the edges other than e that were incident to u and v. Let G be a graph, we denote G/e or $G.e$ for a graph obtained by contracting an edge e. This resulting graph has one edge less than G. In a graph G, deleting an edge e with endpoints u and v is to remove

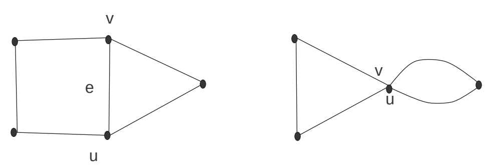


FIGURE 1. Contracting an edge can produce multiples edges or loops

the edge e while leaving the vertices u and v intact. For a graph G,

$G\backslash e$ is a graph defined by:

$$\begin{aligned} V(G) &= V(G); \\ E(G\backslash e) &= E(G)\backslash\{e\}. \end{aligned}$$

(2.2.1)

An edge e is an *isthmus* or a *bridge* or a *cut edge* for the graph G if

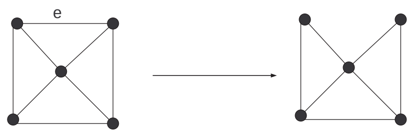


FIGURE 2. Deleting an edge

G is connected and $G\backslash e$ is disconnected. In other words e is an edge whose deletion increases the number of components of G, that is $G\backslash e$ has more components than G. If G is a planar graph then $G\backslash e$ and G/e are planar.

2.2.4. Graph polynomials. To compute the graph polynomials of the class of the graphs we identified is the main object of the thesis. These are: the chromatic, the coboundary and the Tutte polynomials.

2.2.4.1. *notions of chromatic polynomial.* Here we are giving some basic notions of the chromatic polynomials needed in this work. The notions defined in this part are well known in graph theory. We refer the reader to Douglas (1996) for further details. A $\lambda-$*coloring* of a graph G is a labeling $f : V(G) \longrightarrow S$ where $|S| = \lambda$. The labels are the colors, the vertices of one color form a color class. A λ-coloring is *proper* if adjacent vertices have different labels. Given $\lambda \in \mathbb{N}$ and a graph G, the value $\chi(G;\lambda)$ is the number of proper coloring $f : V(G) \longrightarrow [\lambda]$, the λ colors need not all to be used in a coloring f. $\chi(G;\lambda)$ is called the *chromatic polynomial* of G. The following theorem on the chromatic polynomial of a graph G is well known in the literature, see Douglas (1996). It is usually referred to as as the deletion-contraction formula of the chromatic polynomial. Recall that $G\backslash e$ is a minor obtained by deleting e from G and G/e is a minor obtained by contracting e from G.

THEOREM 2.2.3. *If G is a graph and $e \in E(G)$ then*

$$\chi(G;\lambda) = \chi(G\backslash e;\lambda) - \chi(G/e;\lambda).$$

2.2.4.2. *Notions of coboundary polynomial.* The following definitions and facts of coboundary polynomials are well known. We refer the reader to [**?**] for further details. Recall that if $A \subset E$ where E is the edge set of a graph G, then the rank of A, is the number of vertices in A minus the number of connected components of the graph $G(A)$. In other words $r(A) = V(A) - \omega(G\backslash A)$ where $\omega(G\backslash A)$ is the number of connected components of $G(A)$.

DEFINITION 2.2.4. The coboundary polynomial of a graph G, with edge set $E(G)$, is a polynomial in two variables λ and S denoted by $B(G;\lambda,S)$ and is defined as

$$B(G;\lambda,S) = \sum_{A\subseteq E}^{|A|}(S-1)^{|A|}\lambda^{r(E)-r(A)}$$

The coboundary polynomial $B(G;\lambda,S)$ of a graph G is equal to the coboundary polynomial of its cycle matroid $M(G)$. Therefore, the theory of coboundary polynomials for matroids generalizes for graphs.

PROPOSITION 2.2.5. *Let G be a graph, $E(G)$ its edge set and $e \in E(G)$. Then*

(1) *If e is neither a loop nor a coloop of G then*

$$B(G;\lambda,S) = B(G\backslash e;\lambda,S) + (S-1)B(G/e;\lambda,S).$$

(2) *If e is a loop then*

$$B(G;\lambda,S) = SB(G\backslash e;\lambda,S).$$

(3) *If e is a coloop, then*

$$B(G;\lambda,S) = (S+\lambda-1)B(G/e;\lambda,S).$$

Proposition 2.2.5 is refereed to as the deletion and contraction formula for the coboundary polynomial.

2.2.4.3. *Notions on Tutte polynomial.* We begin by giving a formal definition and then, its deletion contraction expression and for further details on the Tutte polynomial we refer the reader to Stephen(2001). Recall that if $A \subset E$ where E is the edge set of a graph $G(E)$, then rank of A, is the number of vertices in A minus the number of connected components of the graph $G(A)$. In other words

$$r(A) = V(G) - \omega(G\backslash A)$$

where $\omega(G\backslash A)$ is the number of connected components of $G(A)$.

Definition 2.2.6. Given a graph G, with edge set E we define a two variable polynomial, the Tutte polynomial,

$$T(G;x,y) = \sum_{A\subseteq E}(x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

The Tutte polynomial of a graph G can be calculated using the the following deletion-contraction method:

(2.2.2)

$$T(G;x,y) = \begin{cases} xT(G/e;x,y) & : \quad if\ e\ is\ a\ isthmus \\ yT(G\backslash e;x,y) & : \quad if\ e\ is\ a\ loop \\ T(G/e;x,y) + T(G\backslash e;x,y) & : \quad otherwise \end{cases}$$

2.2.5. Triangulated ladders: definitions and examples.

DEFINITION 2.2.7. A simple graph is *triangulated* if every cycle of length at least 4, has a chord, that is an edge joining nonadjacent vertices of the cycle. A triangulated graph is also known as a *chordal* graph.

DEFINITION 2.2.8. A graph G is called an n-ladder (see Figure 3,) if it has the vertex set $\{1, 2, 3, \ldots, 2n-1, 2n\}$ and the

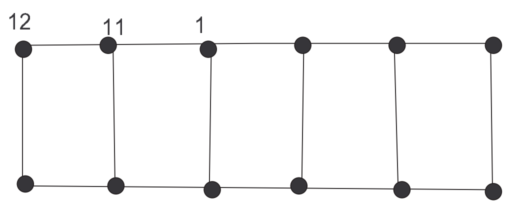


FIGURE 3. A $6 - ladder$

edge set $\{\{1,2\},\{2,3\},\{3,4\},\ldots,\{2n-2,2n-1\},\{2n-1,2n\}\} \cup \{\{1,2n\},\{2,2n-1\},\{3,2n-2\},\ldots,\{n,n+1\}\}$.

EXAMPLE 2.2.9. The following three figures of graphs G_1, G_2 and G_3 are examples of different constructions of triangulated 3-ladders.

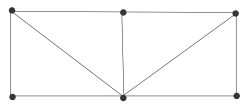


FIGURE 4. G_1

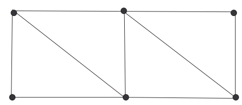


FIGURE 5. G_2

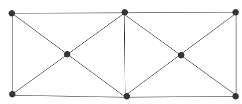


FIGURE 6. G_3

All the diagrams shown in Example 2.2.9 represent different triangulations of a 3-ladder. The graphs G_1, G_2 and G_3 are non isomorphic. Since there are several triangulations of n-ladder, in this thesis we will only consider the triangulation of an n-ladder as shown in Figure 6 by diagram G_3.

DEFINITION 2.2.10. A *triangulated n-ladder* is a graph G, having the following set of vertices $V(G) = \{1, 2, \dots, 3n-1\}$ and edge set $E(G) = E_1(G) \cup E_2(G) \cup E_3(G)$ where $E_1(G) = \{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}\} \cup \{\{n+1, n+2\}, \dots, \{2n-1, 2n\}\}$, $E_2(G) = \{\{1, 2n\}, \{2, 2n-1\}, \dots, \{n, n+1\}\}$ and $E_3(G) = \{\{k, 2n+k\}, \{k+1, 2n+k\}, \{2n-k, 2n+k\}, \{2n+1-k, 2n+k\}\}$ for $k = 1, 2, \dots, n-1$. A triangulated n-ladder is denoted by T_n in this thesis. An edge $e_1 \in E_1(G)$ is called a *bar, an edge $e_2 \in E_2(G)$ is called a rung and an edge $e_3 \in E_3$ is called a spoke.*

Definition 2.2.11. Let T_n be a triangulated n-ladder, then we call n its size, and diagram given in Figure 1 is a graph of a triangulated n-ladder T_n.

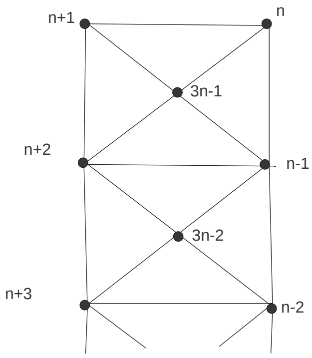
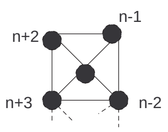
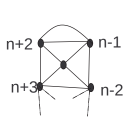
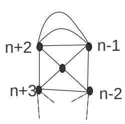


Figure 7. Triangulated n-ladder

2.2.6. Notation. We start by introducing new notation that we will use from this section to the end of the thesis. The following table gives a clear picture of our notation.

Notation 2.2.12. *Here we refer to T_{n-1} as a triangulated $(n-1)$-ladder, that corresponds to a triangulated ladder whose the last rung is $\{(n-1, n+2)\}$. We denote T'_{n-1} for T_{n-1} whose the last rung is replaced by two parallel edges. We denote $T"_{n-1}$ for T_{n-1} whose last rung is replaced by three parallel edges.*

T_{n-1}	T'_{n-1}	T''_{n-1}
		

CHAPTER 3

Edge set, Chromatic and Coboundary Polynomials of a Triangulated ladder.

In this chapter we are proving a proposition on the number of edges that has a triangulated ladder, we set two propositions; one recursive an other explicit for the chromatic polynomial. We close this chapter by computing a recursive expression of the coboundary polynomial of a triangulated ladder.

3.1. Edge set of a triangulated ladder.

The diagram given in Figure 1 is an example of a triangulated n-ladder. In this section we give the number of edges of a triangulated

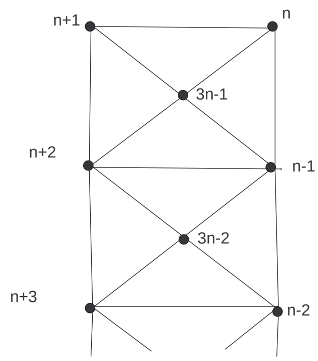


FIGURE 1. Triangulated n-ladder

n-ladder. We need one lemma and the notation 2.2.12 before stating an proving the following proposition on the number of edges of a triangulated n-ladder.

LEMMA 3.1.1. *A triangulated* $2-$*ladder* T_2 *has eight edges.*

PROOF. The diagram in Figure 2 is a graph of a triangulated 2-ladder. It is clear from the diagram that a triangulated 2-ladder is a

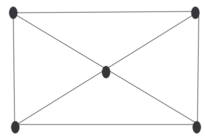


FIGURE 2

4-wheel. But it is well known that an n-wheel has $2n$ edges see Mphako (2002). Hence T_2 has 8 edges □

NOTATION 3.1.2. *Let* E_d *be a subset of edges of* $E(T_n)$ *such that* E_d *consists of four spokes, two bars and one rung as shown in Figure 3. It is clear that* $|E_d| = 7$. *The deletion of* E_d *from* T_n *gives rise to another*

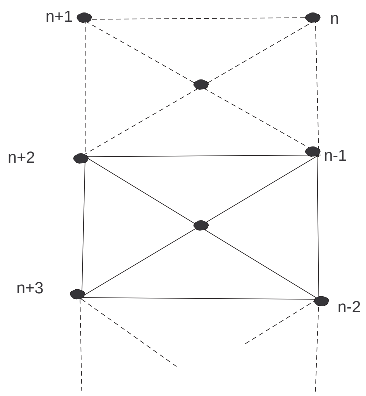


FIGURE 3

triangulated-ladder whose the last rung is {*n-1, n+2*}*: we denote it* T_{n-1}.

Corollary 3.1.3. *It is clear from the previous notation that:*

$$|T_n - E_d| = |T_{n-1}|$$

.

Proposition 3.1.4. *Let T_n be a triangulated n-ladder. Then for $n > 1$,*

$$|E(T_n)| = 7n - 6.$$

Proof. The proof is by induction on n. Let $n = 2$. Then by Lemma 3.1.1, $|E(T_2)| = 8$. Thus

$$|E(T_2)| = 8 = 14 - 6 = (7 \times 2) - 6.$$

Hence it is true for the base case. Assume it is true for some $n = k$. Thus

$$|E(T_k)| = 7k - 6.$$

Now we consider $n = k + 1$. We know that

$$T_{k+1} - E_d = T_k$$

by applying Corollary 3.1.2. Thus

$$|E(T_{k+1})| - |E_d| = |E(T_k)|.$$

But we know from Notation 3.1.2 that $|E_d| = 7$. Thus by induction hypothesis we have

$$\begin{aligned} |E(T_{k+1})| &= |E(T_k)| + |E_d| \\ &= 7k - 6 + 7 \\ &= 7(k+1) - 6 \end{aligned}$$

Therefore the proposition is true for n $\square$

3.2. Chromatic polynomial of a triangulated ladder.

In this section, we give both a recursive and an explicit expression of the chromatic polynomial of a triangulated n-ladder. We use the deletion-contraction formula of the chromatic polynomial as given by Theorem 2.2.3. Note that a graph with a loop can not be colored properly, because we can not make the color of a vertex to be different of itself. Hence we have the following corollary:

COROLLARY 3.2.1. *Let G be a graph with a loop, then $\chi(G;\lambda) = 0$.*

COROLLARY 3.2.2. *Let G be a graph with some parallel edges and G' its simplification. Then*

$$\chi(G;\lambda) = \chi(G';\lambda).$$

Explicit expressions of the chromatic polynomials of certain classes of graphs are known. We give some examples of well known classes of graphs. Recall that a *complete* graph, denoted K_n, is a simple graph on n vertices, whose vertices are pairwise adjacent. A *complement* $\overline{G}$ of a simple graph G is a simple graph with vertex set V(G) defined by $\{u, v\} \in E(\overline{G})$ if and only if $\{u, v\} \notin E(G)$. Hence $\overline{K_n}$ is a graph with n vertices and no edges .

EXAMPLE 3.2.3. The following are some examples:

(1) Let K_n be a complete graph on n vertices, then

$$\chi(K_n;\lambda) = \lambda(\lambda-1)(\lambda-2)\dots(\lambda-n+1).$$

(2) Let t_n be a tree of n vertices, then

$$\chi(t_n;\lambda) = \lambda(\lambda-1)^{n-1}.$$

(3) Let C_n be an n-cycle, then

$$\chi(C_n;\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1).$$

(4) Let $\overline{K_n}$ be the complement of the complete graph K_n, then

$$\chi(\overline{K_n};\lambda) = \lambda^n.$$

Now we are going to state and prove the following lemma:

LEMMA 3.2.4. *Let G be a graph and let G_1 be a graph obtained from G by adding a coloop, then*

$$(G_1;\lambda) = (\lambda-1)\chi(G;\lambda).$$

PROOF. To ease notation, each diagram in Figure 4 represents the chromatic polynomial of the graph. Hence

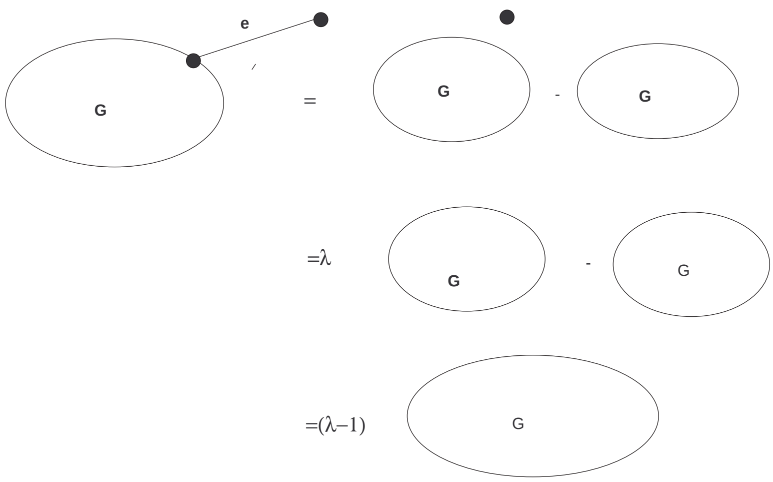


FIGURE 4

$$\chi(G_1;\lambda) = (\lambda-1)\chi(G;\lambda)$$

as required $\square$

NOTATION 3.2.5. *A graph* $\widehat{G}$ *will denote a graph obtained by adding two extra edges, say* f *and* g *to a graph* G *such that* e, f, g *is a three cycle and* $e \in E(G)$.

LEMMA 3.2.6. *Let* G *and* $\widehat{G}$ *be two graphs, and let* e *be an edge in* $E(G)$ *and* f *and* g *be two edges in* $E(\widehat{G}) \backslash E(G)$ *such that the edges* e, f, g *make up a 3-cycle in* $E(\widehat{G})$. *Then*

$$\chi(\widehat{G}; \lambda) = (\lambda - 2)\chi(G; \lambda).$$

PROOF. In this proof we are going to use the deletion contraction formula as shown in Figure 5. To ease notation, each diagram in Figure 5 represents the chromatic polynomial of that graph. Thus by

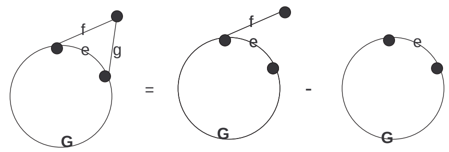


FIGURE 5

applying Lemma 3.2.4, we get

$$\begin{aligned} \chi(\widehat{G}; \lambda) &= (\lambda - 1)\chi(G; \lambda) - \chi(G; \lambda) \\ &= (\lambda - 2)\chi(G; \lambda) \end{aligned}$$

□

COROLLARY 3.2.7. *If* G *is a 3-cycle, then*

$$\chi(G; \lambda) = \lambda(\lambda - 1)(\lambda - 2).$$

LEMMA 3.2.8. *If* G_3 *is a graph as given by Figure 6, then*

$$\chi(G_3; \lambda) = \lambda(\lambda - 1)(\lambda - 2)^2.$$

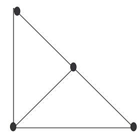


FIGURE 6. The graphG_3

PROOF. By applying Lemma 3.2.6 and Corollary 3.2.7, we get

$$\begin{aligned}\chi(G_3;\lambda) &= (\lambda-2)\chi(C_3;\lambda)\\ &= (\lambda-2)\lambda(\lambda-1)(\lambda-2)\\ &= \lambda(\lambda-1)(\lambda-2)^2\end{aligned}$$

□

LEMMA 3.2.9. *If G_4 is a graph as given by Figure 7, then*

$$\chi(G_4;\lambda) = \lambda(\lambda-1)(\lambda-2)^3.$$

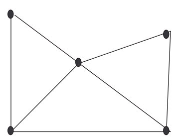


FIGURE 7. The graph G_4

PROOF. By applying Lemma 3.2.6 and Lemma 3.2.8, we get

$$\begin{aligned}\chi(G_4;\lambda) &= (\lambda-2)\chi(G_3;\lambda)\\ &= (\lambda-2)\lambda(\lambda-1)(\lambda-2)^2\\ &= \lambda(\lambda-1)(\lambda-2)^3.\end{aligned}$$

□

The following lemma is used to prove the proposition on the chromatic polynomial for a triangulated n-ladder using the induction principle.

LEMMA 3.2.10. *Let T_2 be a triangulated 2-ladder. Then*

$$\chi(T_2;\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7).$$

PROOF. We use the deletion-contraction formula as shown in Figure 8. To ease notation, the diagrams in Figure 8 represent the chromatic polynomials of the graphs. Recall that a graph G and its simplification have the same polynomial, see Corollary 3.2.2. Hence by applying Lemma 3.2.9, Lemma 3.2.8 and Corollary 3.2.7, we get

$$\begin{aligned}\chi(T_2;\lambda) &= (\lambda-2)^3\lambda(\lambda-1) - (\lambda-2)^2\lambda(\lambda-1) + (\lambda-2)\lambda(\lambda-1)\\ &= (\lambda-2)\lambda(\lambda-1)(\lambda^2-4\lambda+4-\lambda+2+1)\\ &= \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)\end{aligned}$$

□

The following is a recursive formula for the chromatic polynomial of a triangulated n-ladder.

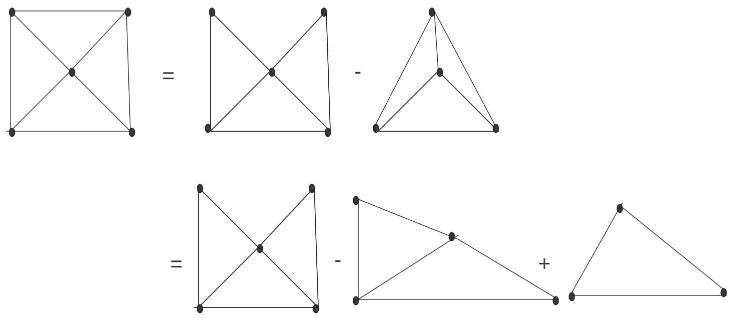


FIGURE 8

PROPOSITION 3.2.11. *Let T_n be a triangulated n-ladder. Then*

$$\chi(T_n;\lambda) = (\lambda-2)(\lambda^2-5\lambda+7)\chi(T_{n-1};\lambda).$$

PROOF. We define a tree diagram by using the deletion-contraction formula at each step. Each graph in the diagram can be replaced by two new graphs. The graph obtained by deletion is characterized by a new subindex 1 added to the former label, and the graph obtained by the contraction is characterized by a new subindex 2 added to the former label. To ease notation the diagrams and their labels in Figure 9 represent the chromatic polynomials of the graphs. From the tree diagram shown in Figure 9, and applying Proposition 2.2.3 and Lemma 3.2.4, we get

$$\chi(T_{n-1};\lambda) = \chi(G1222;\lambda).$$

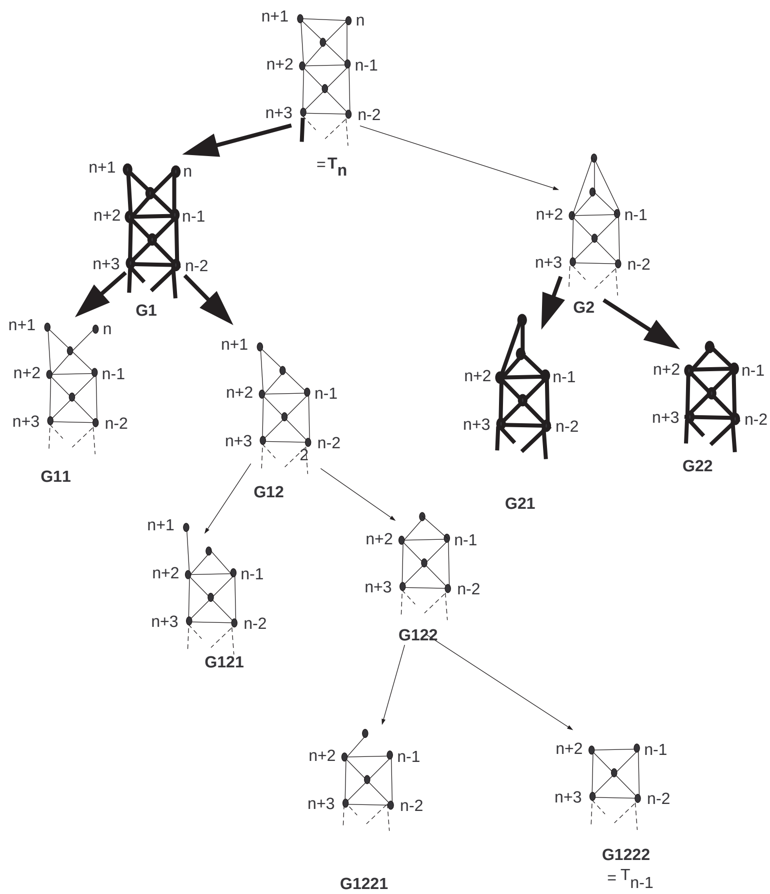


FIGURE 9. A tree diagram for the chromatic polynomial of a T_n.

Gathering equations on one side, to get the chromatic polynomial of graph indexed G1, we have

$$
\begin{aligned}
\chi(G122;\lambda) &= \chi(G1221;\lambda) - \chi(G1222;\lambda) \\
&= (\lambda-1)\chi(G1222;\lambda) - \chi(G1222;\lambda) \\
&= (\lambda-2)\chi(G1222;\lambda) \\
&= (\lambda-2)\chi(T_{n-1};\lambda).
\end{aligned}
$$

Hence

$$\begin{aligned}\chi(G12;\lambda) &= \lambda(G121;\lambda)-\chi(G122;\lambda)\\ &= (\lambda-1)\chi(G122;\lambda)-\chi(G122;\lambda)\\ &= (\lambda-2)\chi(G122;\lambda)\\ &= (\lambda-2)^2\chi(T_{n-1};\lambda).\end{aligned}$$

Thus

$$\begin{aligned}\chi(G1;\lambda) &= \chi(G11;\lambda)-\chi(G12;\lambda)\\ &= (\lambda-1)\chi(G12;\lambda)-\chi(G12;\lambda)\\ &= (\lambda-2)\chi(G12;\lambda)\\ &= (\lambda-2)^3\chi(T_{n-1};\lambda).\end{aligned}$$

Gathering equations on the other side, to get the chromatic polynomial of graphs indexed G21 and G22, we have

$$\chi(G21;\lambda)=\chi(G12;\lambda)=(\lambda-2)^2\chi(T_{n-1};\lambda).$$

$$\chi(G22;\lambda)=\chi(G122;\lambda)=(\lambda-2)\chi(T_{n-1};\lambda).$$

Going back to Figure 9 and applying Proposition 2.2.3, we get

$$\begin{aligned}\chi(T_n;\lambda) &= \chi(G1;\lambda)-\chi(G2;\lambda)\\ &= \chi(G1;\lambda)-\chi(G21;\lambda)+\chi(G22;\lambda)\\ &= (\lambda-2)^3\chi(T_{n-1};\lambda)\lambda)\\ &- (\lambda-2)^2\chi(T_{n-1};\lambda)+(\lambda-2)\chi(T_{n-1};\lambda)\\ &= (\lambda-2)[(\lambda-2)^2-(\lambda-2)+1]\chi(T_{n-1};\lambda)\\ &= \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)\chi(T_{n-1};\lambda)\end{aligned}$$

□

This is one of the main results of this chapter

PROPOSITION 3.2.12. *Let T_n be a triangulated n-ladder for $n > 1$. Then*

$$\chi(T_n;\lambda) = \lambda(\lambda-1)(\lambda-2)^{n-1}(\lambda^2-5\lambda+7)^{n-1}.$$

PROOF. We use induction principle on n to prove this proposition. The base case for $n = 2$ is given by the Lemma 3.2.10. That is

$$\chi(T_2;\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7).$$

Hence the proposition is true for the base case. Let us assume that the assumption is true for some $n = k$ and $k > 2$. Then

$$\chi(T_k;\lambda) = \lambda(\lambda-1)(\lambda-2)^{k-1}(\lambda^2-5\lambda+7)^{k-1}.$$

Now consider the case when $n = k+1$. We apply Proposition 3.2.11 to get

$$\chi(T_{k+1};\lambda) = (\lambda-2)(\lambda^2-5\lambda+7)\chi(T_k;\lambda).$$

Now by induction hypothesis we get

$$\begin{aligned}\chi(T_{k+1};\lambda) &= (\lambda-2)(\lambda^2-5\lambda+7)\chi(T_k;\lambda)\\ &= (\lambda-2)(\lambda^2-5\lambda+7)[\lambda(\lambda-1)(\lambda-2)^{k-1}(\lambda^2-5\lambda+7)^{k-1}]\\ &= \lambda(\lambda-1)(\lambda-2)^{k}(\lambda^2-5\lambda+7)^{k}.\end{aligned}$$

Hence the proposition is true for any n □

3.3. Coboundary polynomial of a triangulated ladder.

In this section, we develop a tree diagram for calculating the coboundary polynomial of a triangulated n-ladder. We then, give a recursive method of computing the coboundary polynomial of a triangulated n-ladder. Finally establish the relationship existing between the the chromatic polynomial and the coboundary polynomial of a triangulated n-ladder. Proposition 2.2.5 is refereed to as the deletion and contraction formula for the coboundary polynomial.

3.3.1. Tree diagram of the coboundary polynomial of T_n. The tree diagram we are going to build, is shown in Figure 10. We start with a tree diagram of T_n. Each graph decomposes in two new graphs according to the deletion and contraction principle. We label the new graph by adding an index 1 to the former label, if this graph is resulting from the deletion and we label the new graph by adding an index 2 to the former label, if this graph is resulting from the contraction. Given that the contraction and deletion formula reduces the number of edges in T_n, and focusing uniquely on edges that are neither loops nor coloops. This process ends once we reach graphs made of T_{n-1} and possibly, with loops or coloops. Our intention is to deduct a recursive relation linking the coboundary polynomial of T_n to that of T_{n-1}. This approach, gives rise to the recursion of the coboundary polynomial of T_n with three terms namely T_{n-1}, T'_{n-1} and $T"_{n-1}$ up to loop class and coloop class. For clarity, we build the same tree diagram without graphs, as shown in Figure 10 and Figure 11. We circle the corresponding labels by a single circle for graphs having T_{n-1}, double circle for graphs having T'_{n-1} and bold circle for graphs having $T"_{n-1}$. Since our tree diagram has twenty-four endpoints, $B(T_n; \lambda, S)$ has twenty four terms. After then, we are going to give an example showing how to get the coefficient $(S-1)^k$ in each term of the polynomial $B(T_n; \lambda, S)$ for example, the term corresponding to the graph $T211212$. We will follow the bold line in Figure 11.

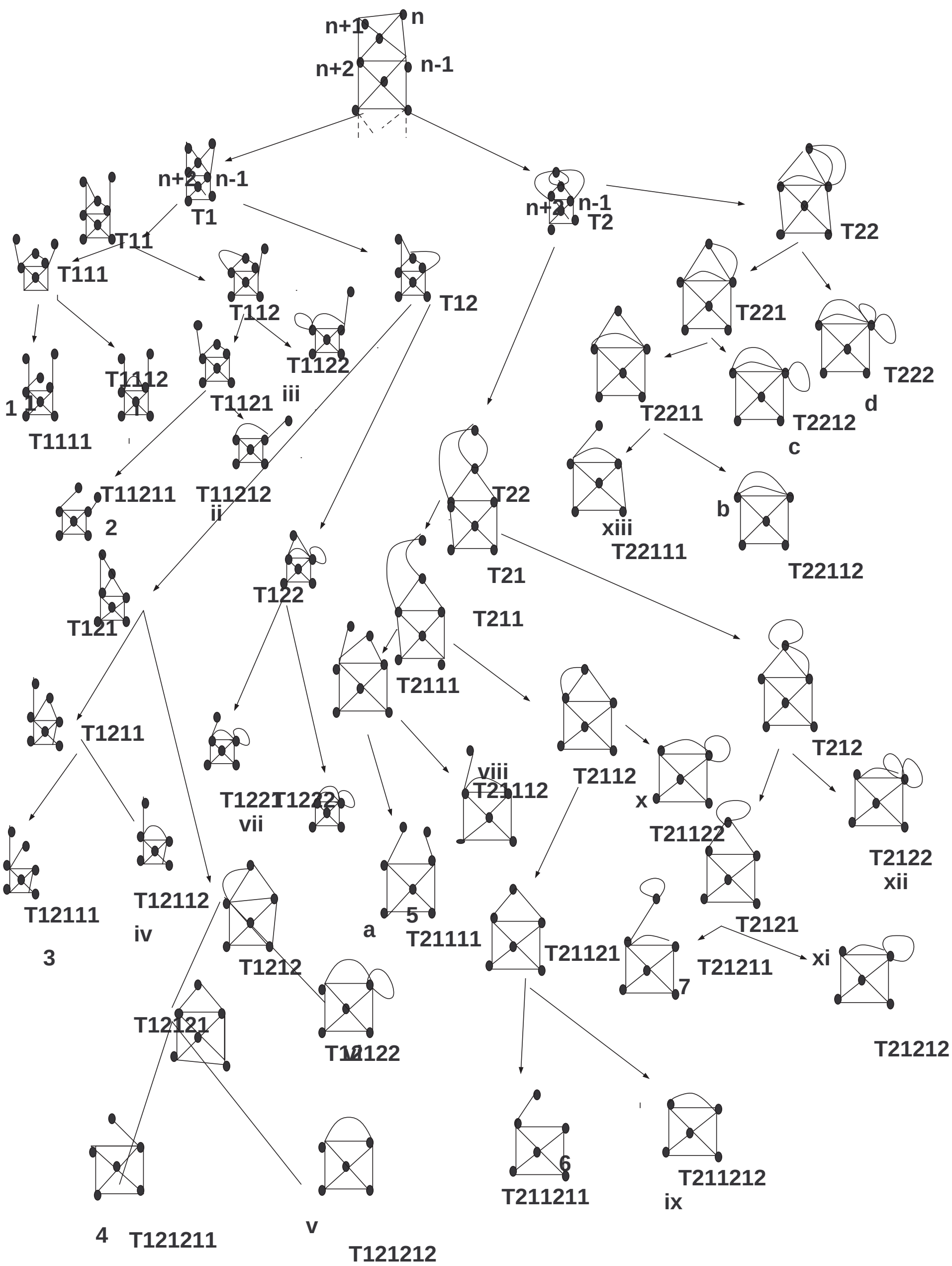


Figure 10. Deletion and contraction of the graph T_n

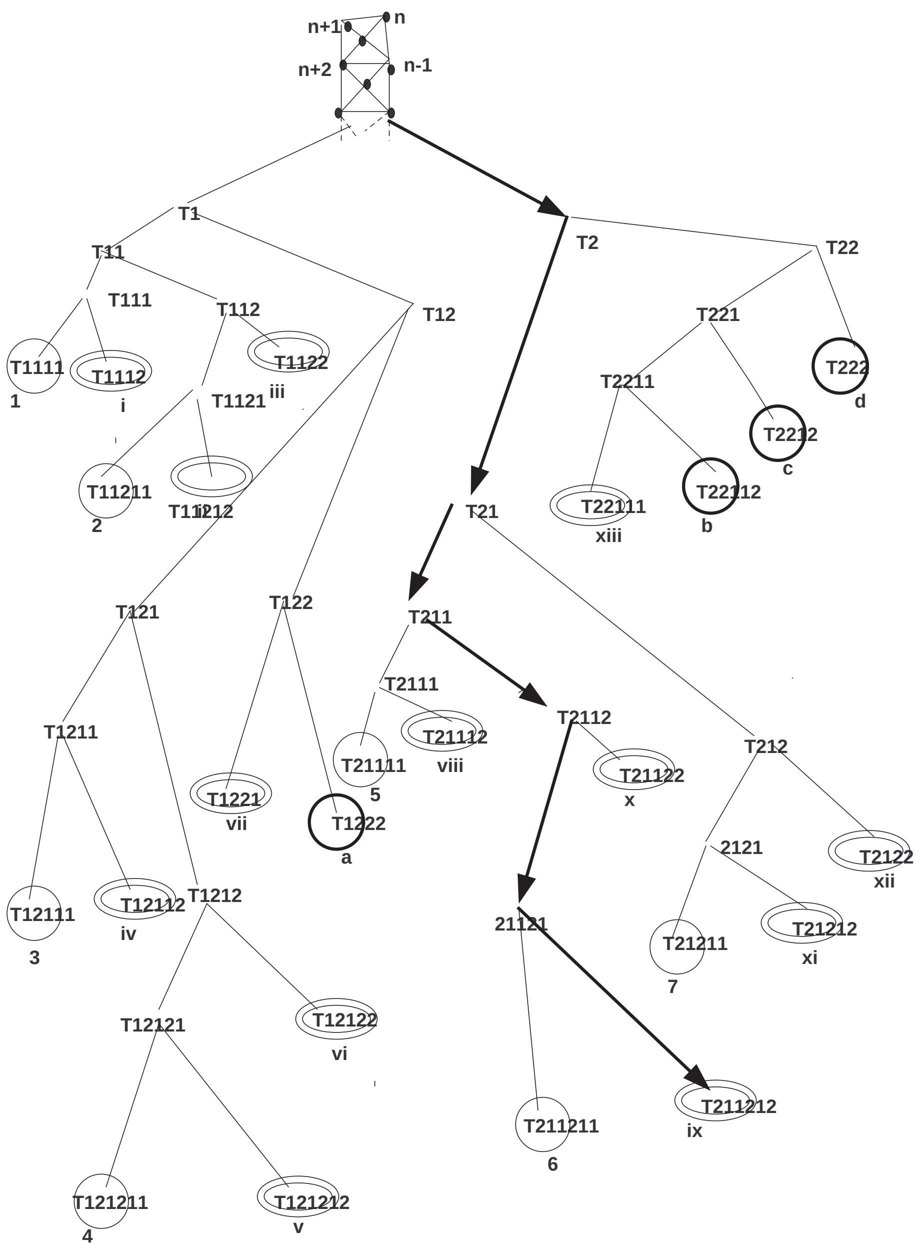


Figure 11. Tree diagram for T_n

Example 3.3.1.

$$
\begin{aligned}
B(T_n;\lambda,S) &= B(T1;\lambda,S) + (S-1)\mathbf{B(T2;\lambda,S)} \\
&= B(T1;\lambda,S) + (S-1)\mathbf{B(T21;\lambda,S)} \\
&+ S-1)^2 B(T22;\lambda,S) \\
&= B(T1;\lambda,S) + (S-1)\mathbf{B(T211;\lambda,S)} \\
&+ (S-1)^2 B(T212;\lambda,S) + (S-1)^2 B(T22;\lambda,S) \\
&= B(T1;\lambda,S) + (S-1)(2111;\lambda,S) \\
&+ (S-1)^2\mathbf{B(T2112;\lambda,S)} \\
&+ (S-1)^2 B(T212;\lambda,S) + (S-1)^2 B(T22;\lambda,S) \\
&= B(T1;\lambda,S) + (S-1)(2111;\lambda,S) \\
&+ (S-1)^2\mathbf{B(T21121;\lambda,S)} \\
&+ (S-1)^3 B(T21122;\lambda,S) + (S-1)^2 B(T212;\lambda,S) \\
&+ (S-1)^2 B(T22;\lambda,S) \\
&= B(T1;\lambda,S) + (S-1)(2111;\lambda,S) \\
&+ (S-1)^2 B(T211211;\lambda,S) + (S-1)^3\mathbf{B(T211212;\lambda,S})) \\
&+ (S-1)^3 B(T21122;\lambda,S) \\
&+ (S-1)^2 B(T212;\lambda,S) + (S-1)^2 B(T22;\lambda,S).
\end{aligned}
$$

Now, we observe that the endpoints of our tree diagram represent the terms of $B(T_n;\lambda,S)$. Each term has a factor $(S-1)^k$ where k is the number of indices 2 in the labeling of the graph at that point. Recall that index 2 correspond to the operation of contraction. Following the deletion and contraction formula again, recall that the presence of a loop gives a factor S and the presence of a coloop gives a factor

$(S+\lambda-1)$. According to the classification of our graphs in three categories, we can split $B(T_n;\lambda,S)$ in three polynomials. We group all terms in $B(T_{n-1};\lambda,S)$ and denote them by

$$\Psi_1(\lambda,S)=\varphi_1(\lambda,S)B(T_{n-1};\lambda,S).$$

We group all terms in $B(T'_{n-1};\lambda,S)$ and denote them by

$$\Psi_2(\lambda,S)=\varphi_2(\lambda,S)B(T'_{n-1};\lambda,S).$$

Finally we group all terms in $B(T"_{n-1};\lambda,S)$ and denote them by

$$\Psi_3(\lambda,S)=\varphi_3(\lambda,S)B(T''_{n-1};\lambda,S).$$

Hence we can express the coboundary polynomial of T_n as a sum of $\Psi_1(\lambda,S)$, $\Psi_2(\lambda,S)$ and $\Psi_3(\lambda,S)$ as follows:

$$B(T_n;\lambda,S) \quad = \quad \Psi_1(\lambda,S)+\Psi_2(\lambda,S)+\Psi_3(\lambda,S). \tag{3.3.1}$$

Now we collect the terms of $B(T_{n-1};\lambda,S)$ from the tree diagrams in Figure 10 and Figure 11 and summarize them in the table form as shown in Figure 12. We now state and prove a lemma on the expression of $\Psi_1(\lambda,S)$.

LEMMA 3.3.2. *Let T_n be a triangulated n-ladder and let $B(T_{n-1};\lambda,S)$ be the coboundary polynomial of T_{n-1}. Then*

$$\begin{aligned}\Psi_1(\lambda,S) &= [(S+\lambda-1)^3+3(S+\lambda-1)^2(S-1)\\ &+ S(S+\lambda-1)(S-1)^2+2(S+\lambda-1)(S-1)^2]B(T_{n-1};\lambda,S),\end{aligned}$$

where $\Psi_1(\lambda,S)$ is a polynomial in λ and S.

PROOF. Recall that $\Psi_1(\lambda,S)=\varphi_1(\lambda,S)B(T_{n-1};\lambda,S)$ and refer to the following table which summarizes the result in Figure 12. Hence the required result $\square$

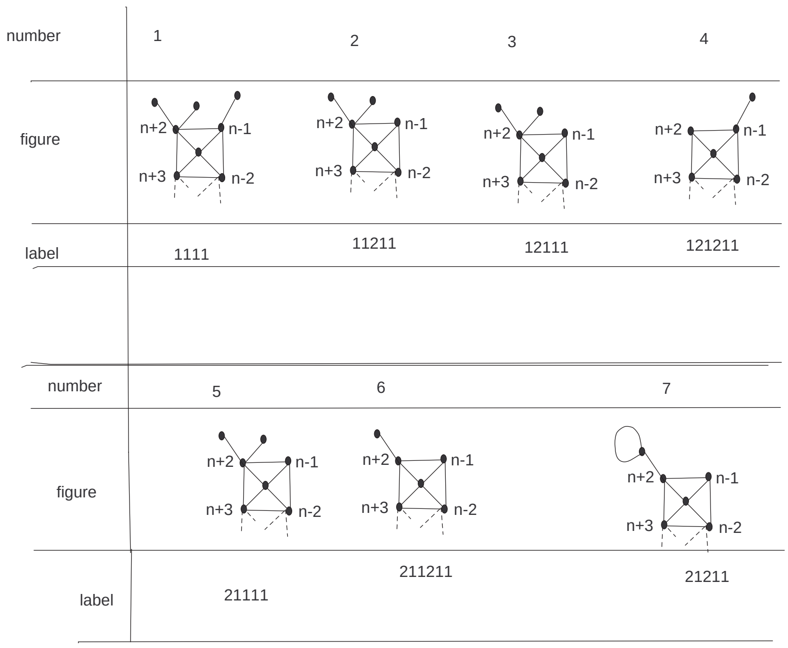


FIGURE 12. Table for $B(T_{n-1}; \lambda, S)$

NOTATION 3.3.3. *We use the following notation in the next tableau.*
I= graph number.
II= number of terms.
III= number of index 2.
IV= number of loops.
V=number of coloops ’

I	II	III	Iv	V	corresponding term
1	1	0	0	3	$1(S+\lambda-1)^3$
2,3,5	3	1	0	2	$3(S+\lambda-1)^2(S-1)$
4,6	2	2	0	1	$2(S+\lambda-1)(S-1)^2$
7	1	2	1	1	$1S(S+\lambda-1)(S-1)^2$

Now we collect the terms of $B(T'_{n-1};\lambda,S)$ from the tree diagrams in Figure 10 and Figure 11 and summarize them in the table form as shown in Figure 13. We now state and prove a lemma on the expression of $\Psi_2(\lambda,S)$.

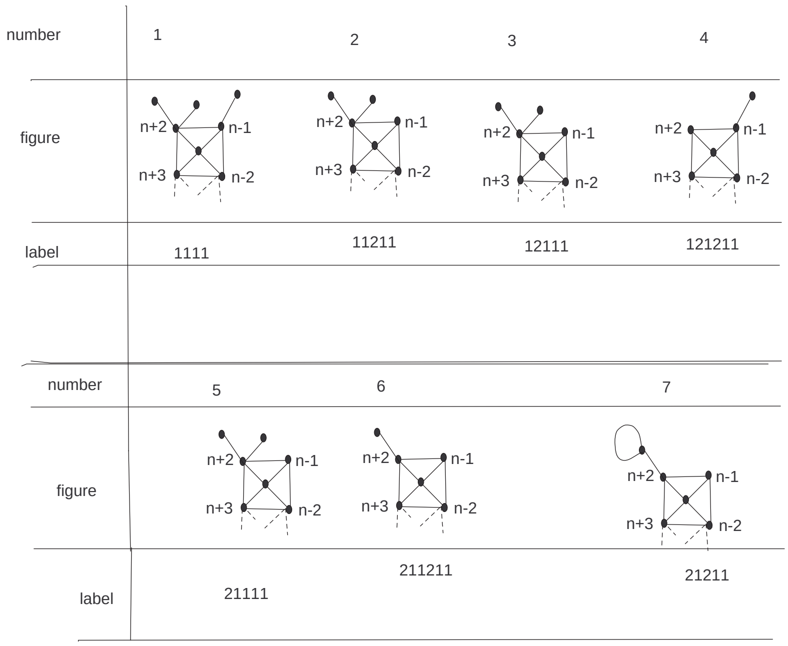


FIGURE 13. Table for $B(T'_{n-1};\lambda,S)$

LEMMA 3.3.4. *Let T_n be a triangulated n-ladder and let $B(T'_{n-1};\lambda,S)$ be the coboundary polynomial of T'_{n-1}. Then*

$$\begin{aligned}\Psi_2(\lambda,S) &= [(S+\lambda-1)^2(S-1)+4(S+\lambda-1)(S-1)^2\\ &+ 2S(S+\lambda-1)(S-1)^2+3S(S-1)^3+S^2(S-1)^3\\ &+ 2(S-1)^3]B(T'_{n-1};\lambda,S),\end{aligned}$$

where $\Psi_2(\lambda,S)$ is a polynomial in λ and S.

PROOF. Recall that $\Psi_2(\lambda, S) = \varphi_2(\lambda, S)B(T'_{n-1}; \lambda, S)$ and refer to the following table which summarizes the result in Figure 13. Hence the result $\square$

NOTATION 3.3.5. *We use the following notation in the next tableau.*
I= graph number.
II=number of terms.
III=number of index 2.
IV= number of loops.
V= number of coloops.

I	II	III	IV	V	corresponding term
i	1	1	0	2	$(S+\lambda-1)^2(S-1)$
ii,iv,viii,xiii	4	2	0	1	$4(S+\lambda-1)(S-1)^2$
iii,vii	2	2	1	1	$2S(S+\lambda-1)(S-1)^2$
vi,x,xi	3	3	1	0	$3S(S-1)^3$
xii	1	3	2	0	$1S^2(S-1)^3$
v,ix	2	3	0	0	$2(S-1)^3$

Now we collect the terms of $B(T''_{n-1}; \lambda, S)$ from the tree diagrams in Figure 10 and Figure 11 and summarize them in the table form as shown in Figure 14. We now state and prove a lemma on the expression of $\Psi_3(\lambda, S)$.

LEMMA 3.3.6. *Let T_n be a triangulated n-ladder and let $B(T"_{n-1}; \lambda, S)$ be the coboundary polynomial of $T"_{n-1}$. Then*

$$\Psi_3(\lambda, S) = [(S-1)^3 + 2S(S-1)^3 + S^2(S-1)^3]B(T''_{n-1}; \lambda, S)$$

where $\Psi_3(\lambda, S)$ is a polynomial in λ and S.

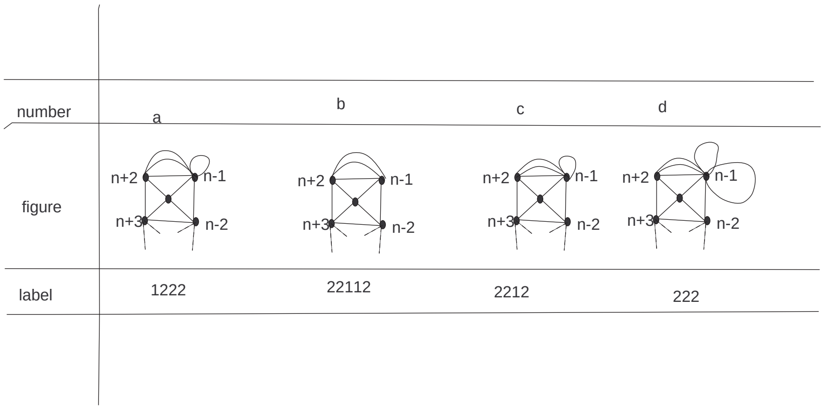


FIGURE 14. Table for $B(T"_{n-1}; \lambda, S)$

PROOF. Recall that $\Psi_3(\lambda, S) = \varphi_3(\lambda, S)B(T"_{n-1}; \lambda, S)$ and refer to the following table which summarizes the result in Figure 14. Hence the result

NOTATION 3.3.7. *We use the following notation in the next tableau.*
I= graph number.
II=number of terms.
III=number of index 2*.*
IV= number of loops.
V= number of coloops.

□

I	II	III	IV	V	corresponding term
d	1	3	2	0	$S^2(S-1)^3$
a,c	2	3	1	0	$2S(S-1)^3$
b	1	3	0	0	$(S-1)^3$

Now we are able to state and prove one of the main results of this chapter. Recall that T_{n-1} is a triangulated $(n-1)$-ladder, T'_{n-1} is a triangulated $(n-1)$-ladder whose last rung is replaced by 2 parallel edges and $T"_{n-1}$ is a triangulated $(n-1)$-ladder whose last rung is replaced by 3 parallel edges.

PROPOSITION 3.3.8. *Let T_n be a triangulated n-ladder and $B(T_n;\lambda,S)$ be the coboundary polynomial of T_n. Then*

$$\begin{aligned} B(T_n;\lambda,S) &= [(S+6)(S-1)^3+(\lambda S+11\lambda)(S-1)^2\lambda) \\ &+ 6\lambda^2(S-1)+\lambda^3]B(T_{n-1};\lambda,S) \\ &+ [(S^2+5S+7)(S-1)^3+(2S\lambda+6\lambda)(S-1)^2\lambda) \\ &+ \lambda^2(S-1)]B(T'_{n-1};\lambda,S) \\ &+ [(S^2+2S+1)(S-1)^3]B(T"_{n-1};\lambda,S). \end{aligned}$$

PROOF. Recall that the coboundary polynomial of T_n can be written as $B(T_n;\lambda,S)=\Psi_1(\lambda,S)+\Psi_2(\lambda,S)+\Psi_3(\lambda,S)$ by Equation 3.3.1. Now we substitute $\Psi_1(\lambda,S)$ with the formula in Lemma 3.3.2, $\Psi_2(\lambda,S)$ with the formula in Lemma 3.3.4 and $\Psi_3(\lambda,S)$ with the formula in Lemma 3.3.6 to get:

$$\begin{aligned} B(T_n;\lambda,S) &= \Psi_1(\lambda,S)+\Psi_2(\lambda,S)+\Psi_3(\lambda,S) \\ &= [(S+\lambda-1)^3+3(S+\lambda-1)^2(S-1) \\ &+ S(S+\lambda-1)(S-1)^2 \\ &+ 2(S+\lambda-1)(S-1)^2]B(T_{n-1};\lambda,S) \\ &+ [(S+\lambda-1)^2(S-1)+4(S+\lambda-1)(S-1)^2 \\ &+ 2S(S+\lambda-1)(S-1)^2 \\ &+ 3S(S-1)^3+S^2(S-1)^3+2(S-1)^3]B(T'_{n-1};\lambda,S) \\ &+ [(S-1)^3+2S(S-1)^3+S^2(S-1)^3]B(T''_{n-1};\lambda,S). \end{aligned}$$

Now expanding and rearranging in power of $(S-1)$, we get

$$
\begin{aligned}
B(T_n;\lambda,S) &= [(S+6)(S-1)^3S \\
&+ (\lambda 11\lambda)(S-1)^2+6\lambda^2(S-1)S \\
&+ \lambda^3]B(T_{n-1};\lambda,S) \\
&+ [(S^2+5S+7)(S-1)^3+(2S\lambda+6\lambda)(S-1)^2 \\
&+ \lambda^2(S-1)]B(T'_{n-1};\lambda,S) \\
&+ [(S^2+2S+1)(S-1)^3]B(T''_{n-1};\lambda,S)
\end{aligned}
$$

□

3.3.2. Examples of coboundary polynomials. In this subsection we compute the coboundary polynomial of a triangulated 2-ladder using deletion and contraction formula. Then we calculate the same polynomial using Proposition 2.2.5 and show that the two are equal verifying our result.

LEMMA 3.3.9. *Let G be a coloop. Then*

$$B(G;\lambda,S)=S+\lambda-1.$$

PROOF. From Proposition 2.2.5 □

LEMMA 3.3.10. *Let G be a graph on two vertices with two parallel edges only. Then*

$$B(G;\lambda,S)=S^2+\lambda-1.$$

PROOF. We refer to Figure 15 for the computation. Hence by ap-

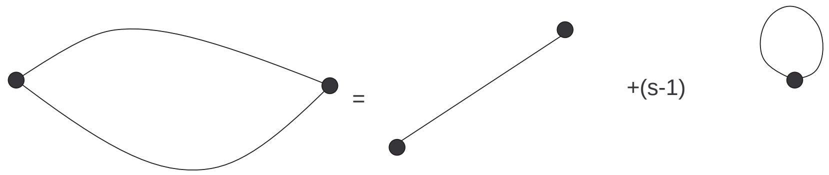


FIGURE 15

plying Proposition 2.2.5 and Lemma 3.3.9 we get

$$\begin{aligned} B(G;\lambda,S) &= S+\lambda-1+S(S-1) \\ &= S+\lambda-1+S^2-S \\ &= S^2+\lambda-1 \end{aligned}$$

□

LEMMA 3.3.11. *Let G be a graph on two vertices with three parallel edges only. Then*

$$B(G;\lambda,S)=S^3+\lambda-1.$$

PROOF. We refer to Figure 16 for the computation. . Hence by

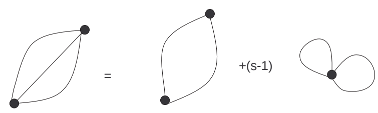


FIGURE 16

applying Proposition 2.2.5 and Lemma 3.3.10 we get

$$\begin{aligned} B(G;\lambda,S) &= S^2+\lambda-1+S^2(S-1) \\ &= S^2+\lambda-1+S^3-S^2 \\ &= S^3+\lambda-1 \end{aligned}$$

□

LEMMA 3.3.12. *Let H_1 be a 3-cycle. Then*

$$B(H_1;\lambda,S) = S^3 + 3(\lambda-1)S + \lambda(\lambda-3) + 2.$$

PROOF. We use the deletion-contraction formula for coboundary polynomial as shown in Figure 17. Hence by applying Proposition 2.2.5

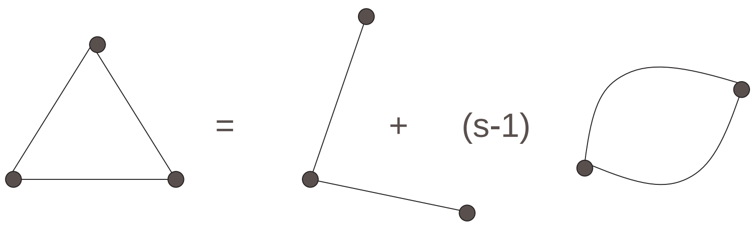


FIGURE 17

and Lemma 3.3.10 we get

$$\begin{aligned} B(H_1;\lambda,S) &= (S+\lambda-1)^2 + (S-1)(S^2+\lambda-1) \\ &= S^3 + 3(\lambda-1)S + \lambda(\lambda-3) + 2 \end{aligned}$$

□

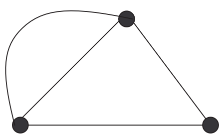


FIGURE 18. H_2

LEMMA 3.3.13. *Let H_2 be the graph shown in Figure 18. Then*

$$B(H_2;\lambda,S) = S^4 + (\lambda-1)S^2 + 2(\lambda-1)S + \lambda(\lambda-3) + 2.$$

PROOF. We use the deletion-contraction formula for the coboundary polynomial as shown in Figure 19. Hence by applying Proposition 2.2.5, Lemma 3.3.10 and Lemma 3.3.12 we get

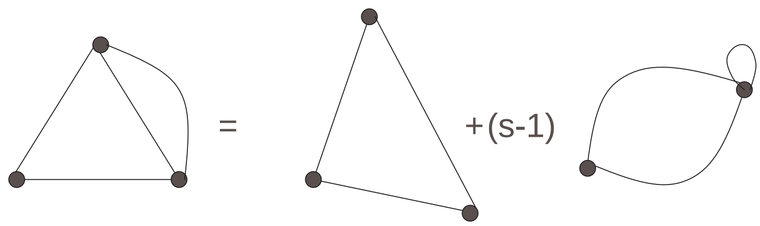


FIGURE 19

$$\begin{aligned} B(H_2;\lambda,S) &= [S^3 + 3(\lambda-1)S + \lambda(\lambda-3) + 2] \\ &+ S(S-1)(S^2+\lambda-1) \\ &= S^4 + (\lambda-1)S^2 + 2(\lambda-1)S + \lambda(\lambda-3) + 2 \end{aligned}$$

□

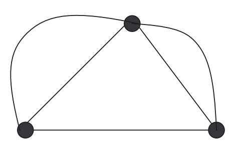


FIGURE 20. H_3

LEMMA 3.3.14. *Let H_3 be the graph shown in Figure 20. Then*

$$B(H_3;\lambda,S) = S^5 + 2(\lambda-1)S^2 + (\lambda-1)S + \lambda(\lambda-3) + 2.$$

PROOF. We use the deletion-contraction formula for the coboundary polynomial as shown in Figure 21. Hence by applying Proposi-

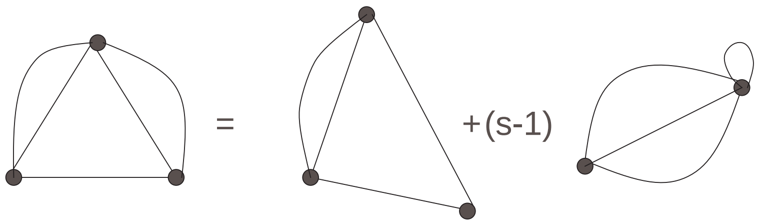


FIGURE 21

tion 2.2.5, Lemma 3.3.11 and Lemma 3.3.13 we get

$$\begin{aligned} B(H_3;\lambda,S) &= [S^4+(\lambda-1)S^2+2(\lambda-1)S+\lambda(\lambda-3)+2] \\ &+ S(S-1)(S^3+\lambda-1) \\ &= S^5+2(\lambda-1)S^2+(\lambda-1)S+\lambda(\lambda-3)+2 \end{aligned}$$

□

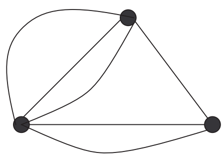


FIGURE 22. H_4

LEMMA 3.3.15. *Let H_4 be the graph shown in Figure 22. Then*

$$B(H_4;\lambda,S)=S^6+(\lambda-1)S^3+(\lambda-1)S^2+(\lambda-1)S+\lambda(\lambda-3)+2.$$

PROOF. We use the deletion-contraction formula for the coboundary polynomial as shown in Figure 23. Hence by applying Proposi-

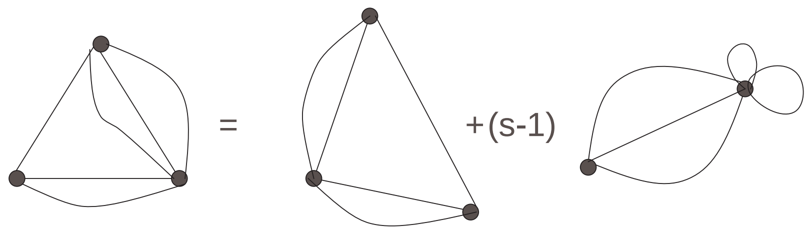


FIGURE 23

tion 2.2.5, Lemma 3.3.11 and Lemma 3.3.14 we get

$$\begin{aligned} B(H_4;\lambda,S) &= [S^5+2(\lambda-1)S^2 \\ &+ (\lambda-1)S+\lambda(\lambda-3)+2]+S^2(S-1)(S^3+\lambda-1) \\ &= S^6+(\lambda-1)S^3+(\lambda-1)S^2+(\lambda-1)S \\ &+ \lambda(\lambda-3)+2 \end{aligned}$$

□

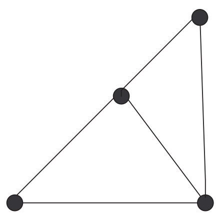


FIGURE 24. H_5

LEMMA 3.3.16. *Let H_5 be the graph shown in Figure 24. Then*

$$B(H_5;\lambda,S)=S^5+2(\lambda-1)S^3+4(\lambda-1)S^2+(\lambda-1)(5\lambda-9)S+(\lambda-2)^2(\lambda-1).$$

PROOF. We use the deletion-contraction formula for the coboundary polynomial as shown in Figure 25. Hence by applying Proposi-

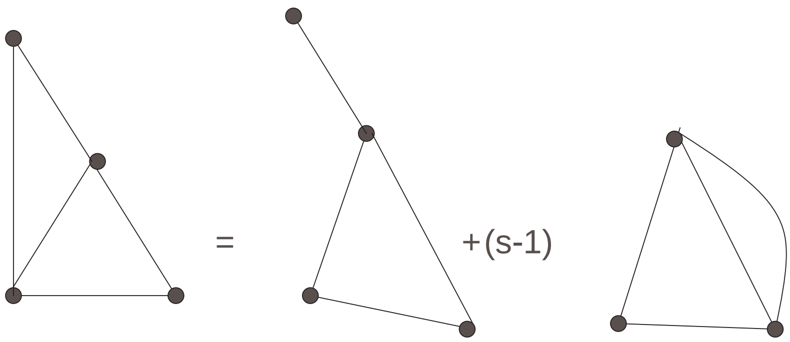


FIGURE 25

tion 2.2.5, Lemma 3.3.12 and Lemma 3.3.13 we get

$$\begin{aligned} B(H_5;\lambda,S) &= [(S+\lambda-1)(S^3 \\ &+ 3(\lambda-1)S+\lambda(\lambda-3)+2] \\ &+ [(S-1)(S^4+(\lambda-1)S^2+2(\lambda-1)S \\ &+ \lambda(\lambda-3)+2)] \\ &= S^5+2(\lambda-1)S^3+4(\lambda-1)S^2 \\ &+ (\lambda-1)(5\lambda-9)S+(\lambda-2)^2(\lambda-1). \end{aligned}$$

□

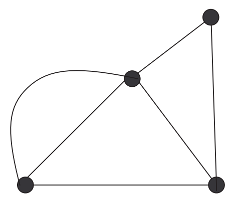


FIGURE 26. H_6

LEMMA 3.3.17. *Let H_6 be the graph shown in Figure 26. Then*

$$B(H_6;\lambda,S) = S^6 + (\lambda-1)S^4 + 3(\lambda-1)S^3 + \lambda(\lambda-1)S^2$$

$$+(4\lambda-7)(\lambda-1)S + (\lambda-2)^2(\lambda-1).$$

PROOF. We use the deletion-contraction formula for the coboundary polynomial as shown in Figure 27. Hence by applying Proposition 2.2.5, Lemma 3.3.13 and Lemma 3.3.16 we get

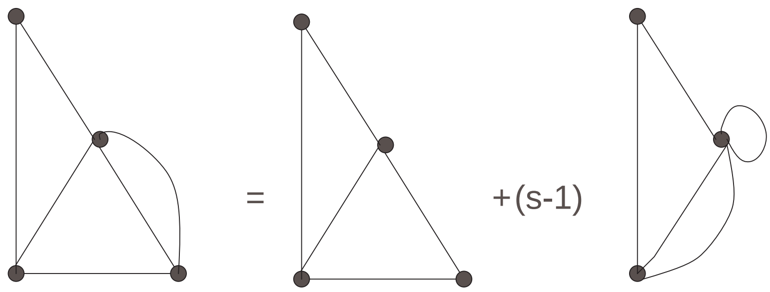


FIGURE 27

$$\begin{aligned}
B(H_6;\lambda,S) &= [S^5 + 2(\lambda-1)S^3 + 4(\lambda-1)S^2 \\
&+ (\lambda-1)(5\lambda-9)s + (\lambda-2)^2(\lambda-1)] \\
&+ [S(S-1)(S^4 + (\lambda-1)S^2 \\
&+ 2(\lambda-1)S + \lambda(\lambda-3)+2)] \\
&= S^6 + (\lambda-1)S^4 + 3(\lambda-1)S^3 + \lambda(\lambda-1)S^2 \\
&+ (4\lambda-7)(\lambda-1)S + (\lambda-2)^2(\lambda-1)
\end{aligned}$$

□

FIGURE 28. H_9

LEMMA 3.3.18. *Let H_9 be the graph shown in Figure 28. Then*

$$
\begin{aligned}
B(H_9;\lambda,S) &= S^7+2(\lambda-1)S^4+3(\lambda-1)S^3 \\
&+ \lambda(\lambda-1)S^2 \\
&+ 5(\lambda-2)(\lambda-1)S+(\lambda-3)\lambda-2)(\lambda-1).
\end{aligned}
$$

PROOF. We use the deletion-contraction formula for the coboundary polynomial as shown in Figure 29. Hence by applying Proposi-

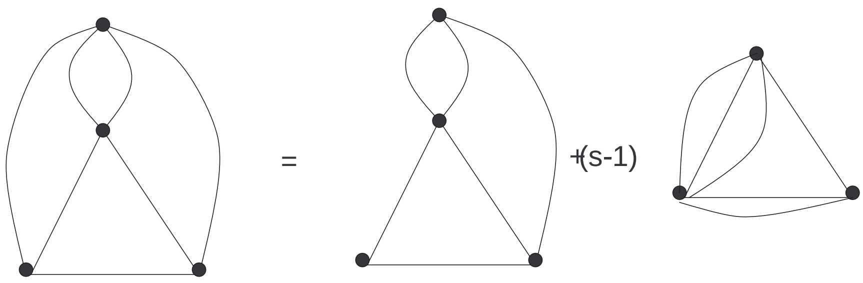


FIGURE 29

tion 2.2.5, Lemma 3.3.17 and Lemma 3.3.15 we get

$$\begin{aligned}
B(H_9;\lambda,S) &= [S^6+(\lambda-1)S^4+3(\lambda-1)S^3 \\
&+ \lambda(\lambda-1)S^2 \\
&+ (4\lambda-7)(\lambda-1)S \\
&+ (\lambda-2)^2(\lambda-1)] \\
&+ (S-1)[S^6+(\lambda-1)S^3+(\lambda-1)S^2+(\lambda-1)S \\
&+ \lambda(\lambda-3)+2] \\
&= S^7+2(\lambda-1)S^4+3(\lambda-1)S^3+\lambda(\lambda-1)S^2 \\
&+ 5(\lambda-2)(\lambda-1)S+(\lambda-3)\lambda-2)(\lambda-1)
\end{aligned}$$

□

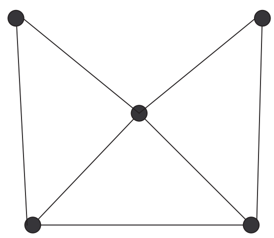
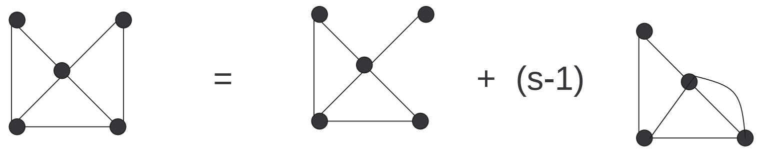


FIGURE 30. H_{10}

LEMMA 3.3.19. *Let H_{10} be the graph shown in Figure 30. Then*

$$\begin{aligned}
B(H_{10};\lambda,S) &= S^7+2(\lambda-1)S^5 \\
&+ 4(\lambda-1)S^4+(3\lambda-1)(\lambda-1)S^3+4(3\lambda-5)(\lambda-1)S^2 \\
&+ (7\lambda-12)(\lambda-2)(\lambda-1)S \\
&+ (\lambda-2)^3(\lambda-1).
\end{aligned}$$

Proof. We use the deletion-contraction formula for the coboundary polynomial as shown in Figure 31. Hence by applying Proposi-



.

Figure 31

tion 2.2.5, Lemma 3.3.16 and Lemma 3.3.17 we get

$$\begin{aligned}
B(H_{10};\lambda,S) &= (S+\lambda-1)[S^5\\
&+ 2(\lambda-1)S^3+4(\lambda-1)S^2\\
&+ (\lambda-1)(5\lambda-9)S+(\lambda-2)^2(\lambda-1)]\\
&+ (S-1)[S^6+(\lambda-1)S^4+3(\lambda-1)S^3+\lambda(\lambda-1)S^2\\
&+ (4\lambda-7)(\lambda-1)S+(\lambda-2)^2(\lambda-1)]\\
&= S^7\\
&+ 2(\lambda-1)S^5+4(\lambda-1)S^4\\
&+ (3\lambda-1)(\lambda-1)S^3\\
&+ 4(3\lambda-5)(\lambda-1)S^2\\
&+ (7\lambda-12)(\lambda-2)(\lambda-1)S+(\lambda-2)^3(\lambda-1)
\end{aligned}$$

□

Now we are able to give the coboundary polynomial for T_2 using deletion-contraction formula. Then we compare it with the coboundary polynomial for T_2 using Proposition 2.2.5.

EXAMPLE 3.3.20. Let T_2 be the triangulated 2-ladder. We use the deletion-contraction formula for the coboundary polynomial as shown in Figure 32. We observe that

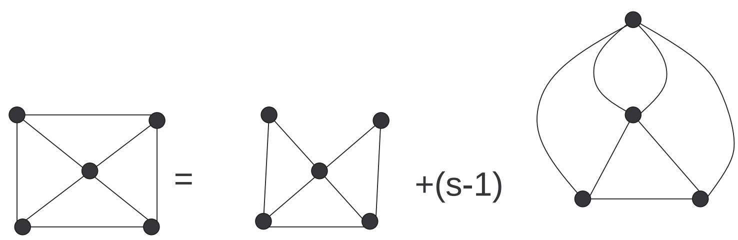


FIGURE 32

$$B(T_2; \lambda, S) = B(H_{10}; \lambda, S) + (S - 1)B(H_9; \lambda, S).$$

But we know $B(H_{10};\lambda,S)$ and $B(H_9;\lambda,S)$ by Lemma 3.3.19 and Lemma 3.3.18 respectively. Hence we substitute to get

$$\begin{aligned}
B(T_2;\lambda,S) &= B(H_{10};\lambda,S)+(S-1)B(H_9;\lambda,S)\\
&= [S^7+2(\lambda-1)S^5+4(\lambda-1)S^4\\
&+ (3\lambda-1)(\lambda-1)S^3+4(3\lambda-5)(\lambda-1)S^2\\
&+ (7\lambda-12)(\lambda-2)(\lambda-1)S\\
&+ (\lambda-2)^3(\lambda-1)]\\
&+ (S-1)[S^7+2(\lambda-1)S^4+3(\lambda-1)S^3\\
&+ \lambda(\lambda-1)S^2\\
&+ 5(\lambda-2)(\lambda-1)S+(\lambda-3)\lambda-2)(\lambda-1)]\\
&= S^8+4(\lambda-1)S^5+5(\lambda-1)S^4\\
&+ 4(\lambda-1)^2S^3+2(8\lambda-15)(\lambda-1)S^2\\
&+ 4(2\lambda-5)(\lambda-2)(\lambda-1)S\\
&+ (\lambda^2-5\lambda+7)(\lambda-2)(\lambda-1).
\end{aligned}$$

Let T_2 be the triangulated 2-ladder. We use the Proposition 2.2.5 to compute the coboundary polynomial of T_2. We get

$$\begin{aligned}
B(T_2;\lambda,S) &= [(S+6)(S-1)^3+(\lambda S+11\lambda)(S-1)^2\\
&+ 6\lambda^2(S-1)+\lambda^3]B(T_1;\lambda,S)\\
&+ [(S^2+5S+7)(S-1)^3+(2S\lambda+6\lambda)(S-1)^2\\
&+ \lambda^2(S-1)]B(T_1';\lambda,S)\\
&+ [(S^2+2S+1)(S-1)^3]B(T"_1;\lambda,S).
\end{aligned}$$

In this case T_1 is a coloop, T_1' is a parallel pair of edges and $T"_1$ is a bunch of three parallel edges. The coboundary polynomials $B(T_1;\lambda,S)$

is given by Lemma 3.3.9, $B(T_1'; \lambda, S)$ is given by Lemma 3.3.10 and $B(T"_1; \lambda, S)$ is given by Lemma 3.3.11. Hence

$$\begin{aligned}
B(T_2; \lambda, S) &= [(S+6)(S-1)^3 + (\lambda S + 11\lambda)(S-1)^2 \\
&+ 6\lambda^2(S-1) + \lambda^3][S+\lambda-1] \\
&+ [(S^2+5S+7)(S-1)^3 + (2S\lambda + 6\lambda)(S-1)^2 \\
&+ \lambda^2(S-1)][S^2+\lambda-1] \\
&+ [(S^2+2S+1)(S-1)^3][S^3+\lambda-1] \\
&= S^8 + 4(\lambda-1)S^5 + 5(\lambda-1)S^4 \\
&+ 4(\lambda-1)^2S^3 + 2(8\lambda-15)(\lambda-1)S^2 \\
&+ 4(2\lambda-5)(\lambda-2)(\lambda-1)S \\
&+ (\lambda^2-5\lambda+7)(\lambda-2)(\lambda-1).
\end{aligned}$$

Thus the coboundary polynomial $B(T_2; \lambda, S)$ using deletion and contraction method is the same as the one found using Proposition 3.3.8.

3.3.3. Chromatic and coboundary polynomials. In this subsection we show the relationship existing between the chromatic polynomial of a triangulated n-ladder and the coboundary polynomial of a triangulated n-ladder. For this reason we need three lemmas, and the main proposition is proved by the use of the induction method.

LEMMA 3.3.21. *Let T_n be a triangulated n-ladder. Then*

$$B(T_n; \lambda, 0) = B(T_n'; \lambda, 0),$$

where T_n' is a graph obtained from T_n by replacing the last rung by two parallel edges.

PROOF. The result follows from the deletion-contraction as shown in Figure 33. To ease notation, each diagram in the Figure 33 represents the coboundary polynomial of that graph □

LEMMA 3.3.22. *Let T_n be a triangulated n-ladder. Then*

$$B(T_n;\lambda,0) = B(T''_n;\lambda,0)$$

where T"$_n$ is a graph obtained from T_n by replacing the last rung by three parallel edges.

PROOF. The result follows from the deletion-contraction as shown in Figure 34. To ease notation, each diagram in the Figure 34 represents the coboundary polynomial of that graph □

LEMMA 3.3.23. *Let $B(T_n;\lambda,S)$ be a coboundary polynomial of the triangulated n-ladder T_n, then*

$$B(T_n;\lambda,0) \;=\; (\lambda-2)(\lambda^2-5\lambda+7)B(T_{n-1};\lambda,0).$$

PROOF. We know by Proposition 3.3.8 that:

$$\begin{aligned}
B(T_n;\lambda,S) &= [(S+6)(S-1)^3+(\lambda S\\
&+ 1\lambda)(S-1)^2+6\lambda^2(S-1)\\
&+ \lambda^3]B(T_{n-1};\lambda,S)+[(S^2+5S+7)(S-1)^3\\
&+ (2S\lambda+6\lambda)(S-1)^2+\lambda^2(S-1)]B(T'_{n-1},\lambda,S)\\
&+ [(S^2+2S+1)(S-1)^3]B(T''_{n-1};\lambda,S).
\end{aligned}$$

If we substitute $S=0$, we get:

$$\begin{aligned}
B(T_n;\lambda,0) &= (\lambda^3-6\lambda^2+11\lambda-6)B(T_{n-1};\lambda,0)\\
&+ (-\lambda^3+6\lambda^2-7)B(T'_{n-1},\lambda,0)\\
&+ (-1)B(T''_{n-1};\lambda,0).
\end{aligned}$$

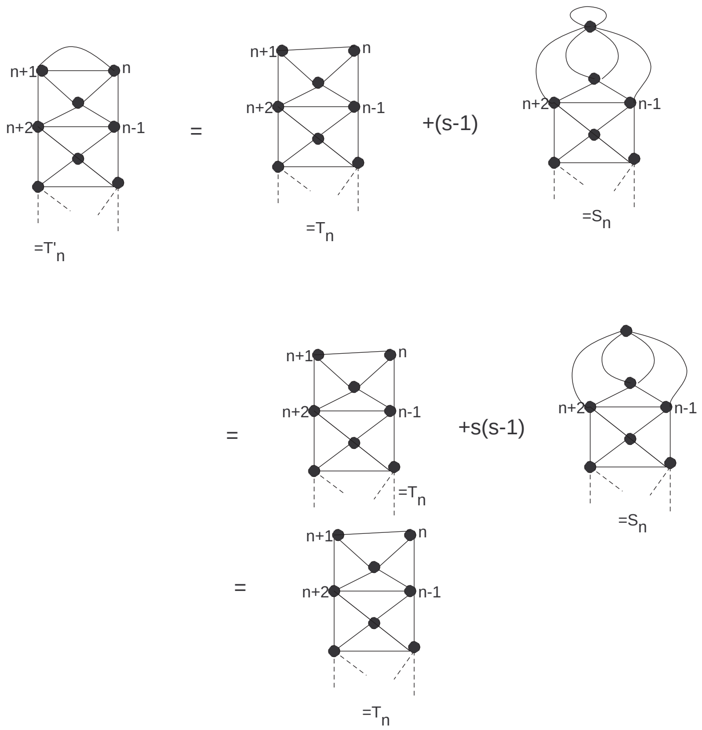


FIGURE 33

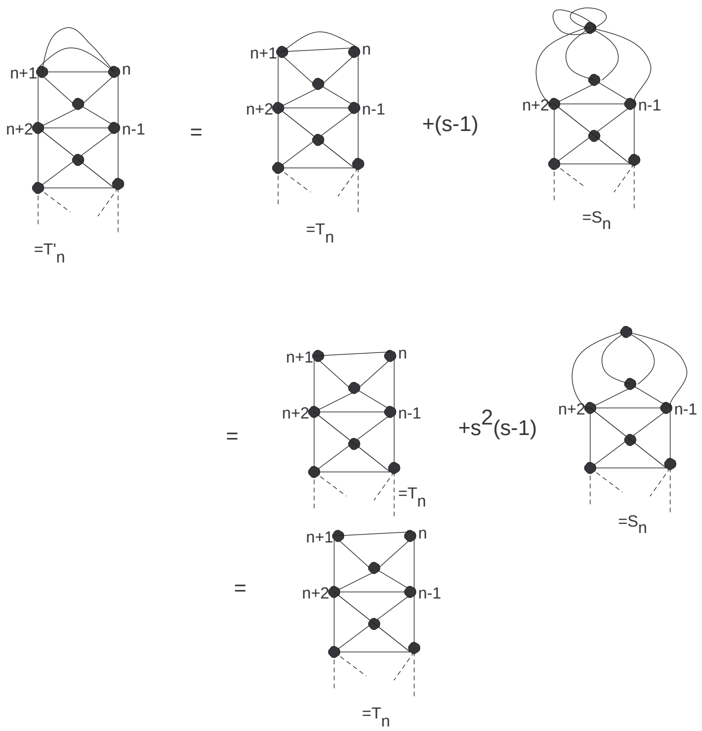


FIGURE 34

But by lemmas [3.3.21 and 3.3.22], we know that:

$$B(T_{n-1};\lambda,0) = B(T'_{n-1};\lambda,0) = B(T''_{n-1};\lambda,0).$$

Hence we get:

$$B(T_n;\lambda,0) \quad = \quad (\lambda-2)(\lambda^2-5\lambda+7)B(T_{n-1};\lambda,0).$$

□

Now we can exhibit the proposition verifying the relationship existing between the coboundary polynomial and the chromatic polynomial of a triangulated ladder.

PROPOSITION 3.3.24. *Let $B(T_n;\lambda,S)$ be a coboundary polynomial of the triangulated n-ladder T_n, then*

$$\chi(T_n;\lambda) = \lambda B(T_n;\lambda,0).$$

PROOF. The proof of this proposition is done by the induction method on the number n. For $n=1$, we know by Lemma 3.3.9 that

$$B(T_1;\lambda,0) = \lambda - 1.$$

and we know that: T_1 is a coloop; a tree on two vertices,

$$\chi(T_1;\lambda) = \lambda(\lambda-1);$$

therefore,

$$\chi(T_1;\lambda) = \lambda(B(T_1;\lambda,0).$$

Hence we proved the base case. Now suppose that the proposition is true for any positive integer k, that is :

$$\chi(T_k;\lambda) = \lambda B(T_k;\lambda,0),$$

and show that the proposition is true for $k+1$. If we multiply both sides of the previous equation by $(\lambda-2)(\lambda^2-5\lambda+7)$, we have:

$$\chi(T_k;\lambda)(\lambda-2)(\lambda^2-5\lambda+7) = \lambda B(T_k;\lambda,0)(\lambda-2)(\lambda^2-5\lambda+7).$$

Respectively, we use Proposition 3.2.11 and Lemma 3.3.23 for the first and the second member of the previous equation and we obtain:

$$\chi(T_{k+1};\lambda) = \lambda B(T_{k+1};\lambda,0).$$

Therefore, we verified the induction hypothesis. Hence, the proposition is true for any positive integer n as required □

CHAPTER 4

Tutte polynomial and link associated to a Triangulated ladder.

In this chapter, we compute the recursive expression of the Tutte polynomials by means of a tree diagram, and we give some examples. We discuss also on links associated to a triangulated ladder, and close it by giving its component number.

4.1. Tutte polynomial for a triangulated n-ladder.

In this section we give a recursive method of the Tutte polynomial of a triangulated n-ladder, using a suitable tree diagram and appropriate tableaux .We give also explicit expressions of the Tutte polynomials of certain graphs, and we verify our result for T_2.

Example 4.1.1. Here, we give some examples of Tutte polynomials of well known classes of graphs.

(1) If t_n is a tree with n vertices, then

$$T(t_n; x, y) = x^{n-1}.$$

(2) If C_n is a cycle with n vertices, then

$$T(C_n; x, y) = x^{n-1} + x^{n-1} + \ldots + x + y.$$

(3) If G is a graph made of n parallel edges, then

$$T(G; x, y) = x + y + \ldots + y^{n-2} + y^{n-1}.$$

4.1.1. Tree diagram of the Tutte polynomial. In this section we are going to give a tree diagram of the Tutte polynomial of a triangulated n-ladder. We start the tree diagram with the graph T_n. Each graph decomposes in two new graphs according to the deletion and contraction relations. We label the new graph by adding an index 1 to the former label, if this graph is resulting from the deletion and we label the new graph by adding an index 2 to the former label, if this graph is resulting from the contraction. Given that the contraction and deletion operations reduce the number of edges and focusing uniquely on edges that are neither loops nor coloops, this process ends once we reach graphs having the structure of T_{n-1} up to parallel class and possibly, with loops or coloops. A general tree diagram of a triangulated n-ladder is given in Figure 1. Our intention is to deduct a recursive relation linking the Tutte polynomial of T_n to that of T_{n-1}. Recall that T'_{n-1} and $T"_{n-1}$ denote a triangulated $(n-1)$-ladder, T_{n-1}, with the last rung being replaced by 2 parallel rungs and 3 parallel rungs respectively. Then this approach, gives the Tutte polynomial of T_n in terms of the Tutte polynomials of T_{n-1}, T'_{n-1} and $T"_{n-1}$ up to simplification. For clarity, we build the same tree diagram of T_n as shown in Figure 1, but this time we replace the graphs corresponding to T_{n-1} by a single circle, corresponding to T'_{n-1} by double circles and corresponding to $T"_{n-1}$ by a bold single circle as shown in Figure 2. The Tutte polynomial, $T(T_n; x, y)$ has twenty four terms since our tree diagram has twenty-four endpoints. This will be simplified later. Hence we can write the Tutte polynomial of T_n as follows:

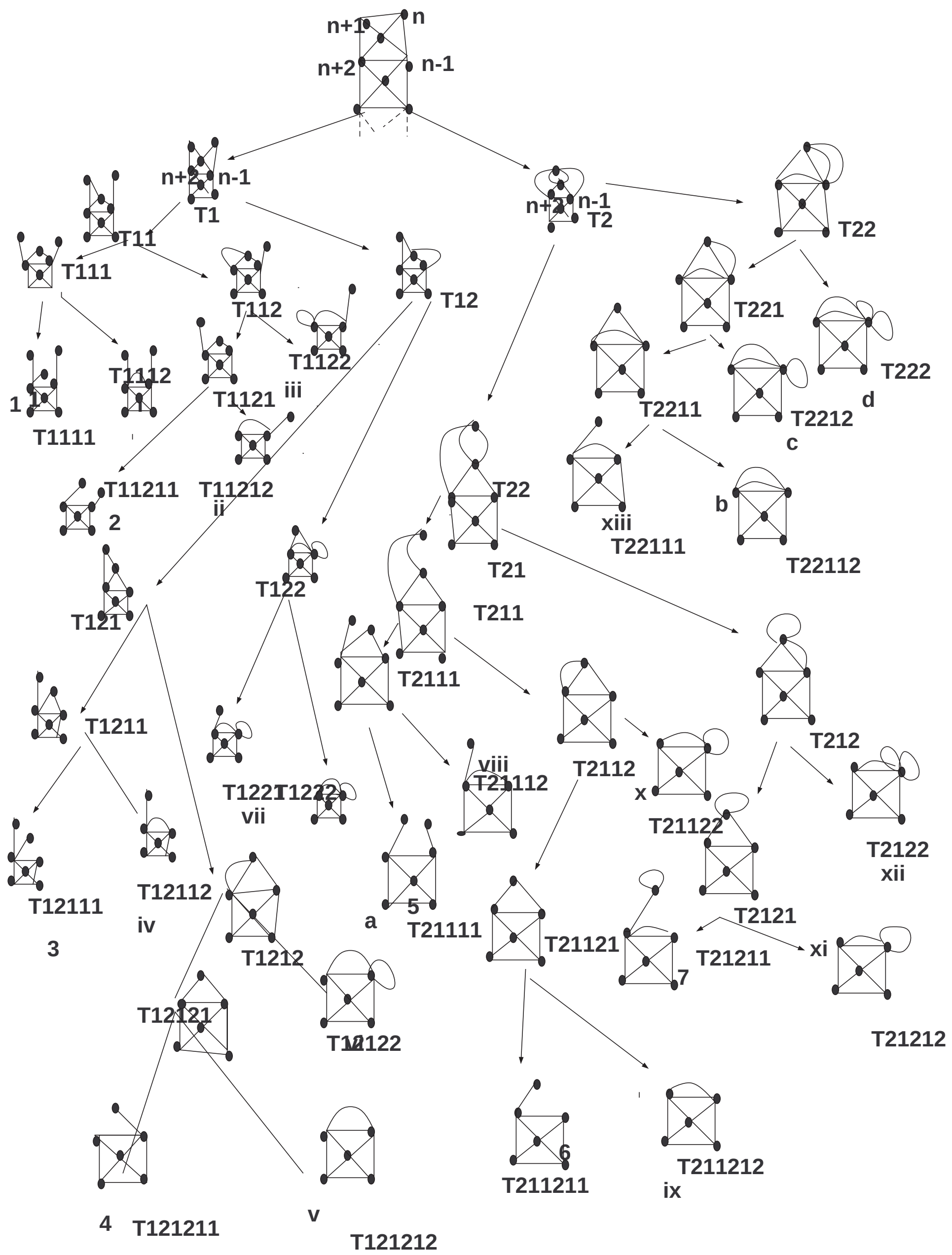


Figure 1. Tree diagram of T_n

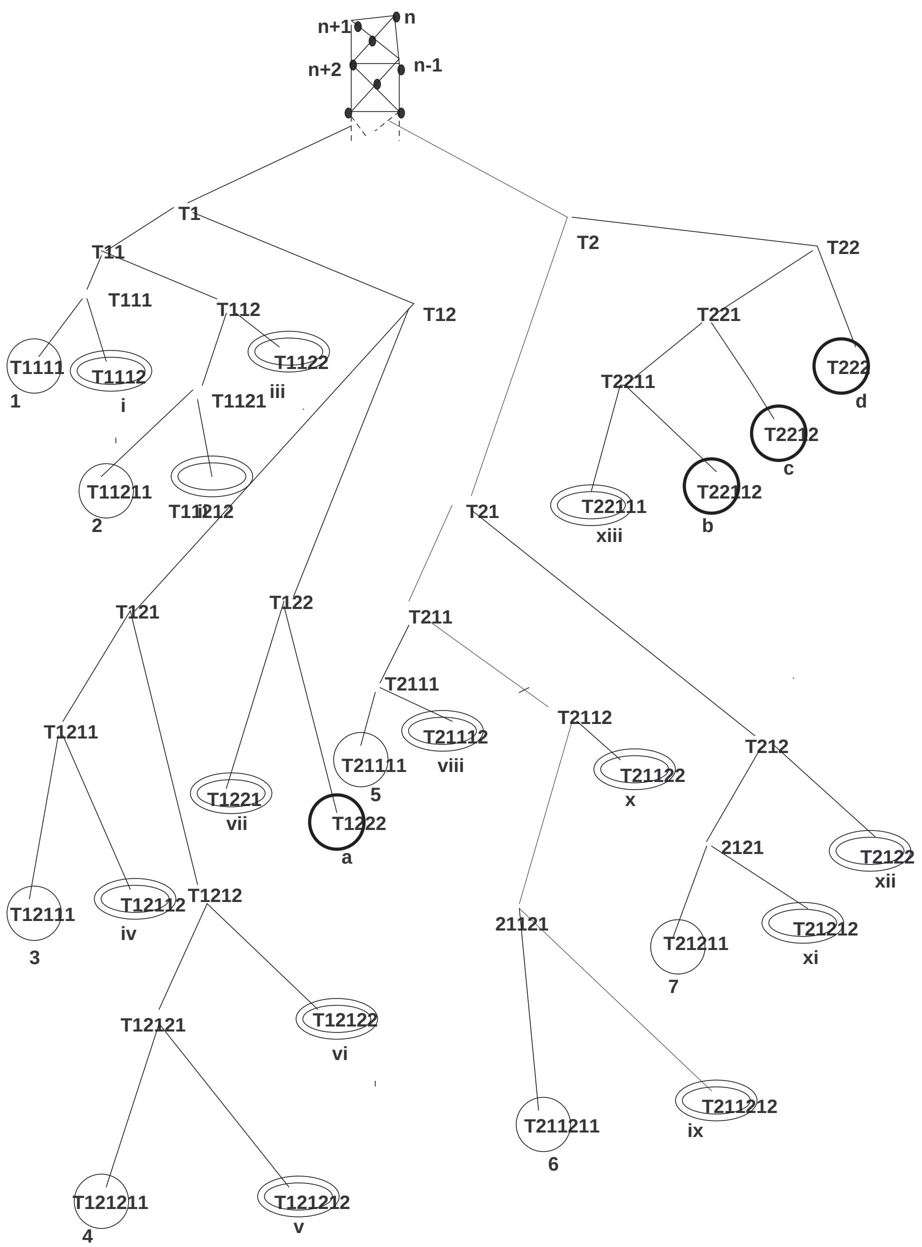


Figure 2. Tree diagram for T_n

$$\begin{aligned}
T(T_n;x,y) &= [T(T1111;x,y)+T(T11211;x,y)+T(T12111;x,y)\\
&+ T(T121211;x,y)+T(T21111;x,y)+T(T211211;x,y)\\
&+ T(T21211;x,y)]+[T(T1112;x,y)+T(T11212;x,y)\\
&+ T(T1122;x,y)+T(T12112;x,y)+T(T1221;x,y)\\
&+ T(T121212;x,y)+T(T12122;x,y)+T(T21112;x,y)\\
&+ T(T22111;x,y)+T(T21122;x,y)+T(T211212;x,y)\\
&+ T(T21212;x,y)+T(T2122;x,y)]\\
&+ [T(T1222;x,y)+T(T22112;x,y)+T(T2212;x,y)\\
&+ T(T222;x,y].
\end{aligned}$$

According to the classification of the simplified minors of T_n in three categories, T_{n-1}, T'_{n-1} and $T"'_{n-1}$ we can write $T(T_n;x,y)$ as a sum of three polynomials. We group all terms in $T(T_{n-1};x,y)$ and denote it by

$$\Psi_1(x,y)=\varphi_1(x,y)(T(T_{n-1};x,y)).$$

Then we group all terms in $T(T'_{n-1};x,y)$ and denote it by

$$\Psi_2(x,y)=\varphi_2(x,y)(T(T'_{n-1};x,y))$$

and finally we group all terms in $(T"'_{n-1};x,y)$ and we denote it by

$$\Psi_3(x,y)=\varphi_3(x,y)(T(T''_{n-1};x,y)).$$

Then we have

$$T(T_n;x,y)=\Psi_1(x,y)+\Psi_2(x,y)+\Psi_3(x,y).$$

The corresponding graphs to T_{n-1}, T'_{n-1} and $T"'_{n-1}$ can be collected in three tables as shown in Figures [3, 4, 5] respectively, highlighting loops and coloops. Following the deletion-contraction formula, we recall that a loop can be deleted and the Tutte polynomial of the minor multiplied by y and similarly for an isthmus where the Tutte polynomial of the minor is multiplied by x. We need the following three lemmas before giving the recursive formula of the Tutte polynomial of a triangulated n-ladder. We begin by gathering terms in $T(T_{n-1}; x, y)$ as shown in Figure 3.

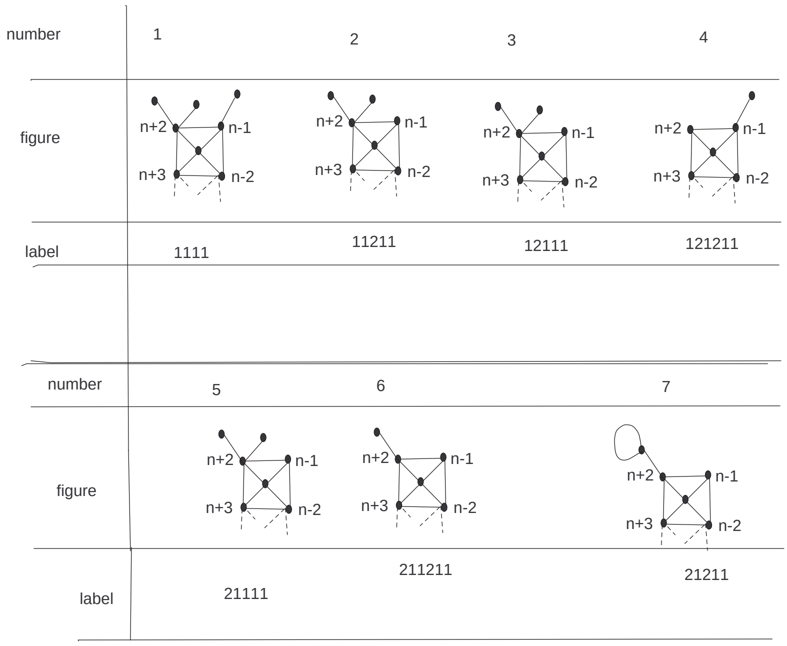


FIGURE 3. Table for T_{n-1}

LEMMA 4.1.2. *Let $\Psi_1(x, y)$ be a two variable polynomial and Let $T(T_{n-1}; x, y)$ be the Tutte polynomial of a triangulated $(n-1)$-ladder. Then*

$$\Psi_1(x, y) = [x^3 + 3x^2 + xy + 2x]T(T_{n-1}; x, y).$$

PROOF. The following table analyzes the result of the terms of $T(T_{n-1}; x, y)$ shown in Figure 3. $\square$

number of terms	with loop	with coloop	corresponding term
1	0	3	$1x^3$
3	0	2	$3x^2$
2	0	1	$2x$
1	1	1	$1xy$

We now gather terms in $T(T'_{n-1}; x, y)$ as shown in Figure 4.

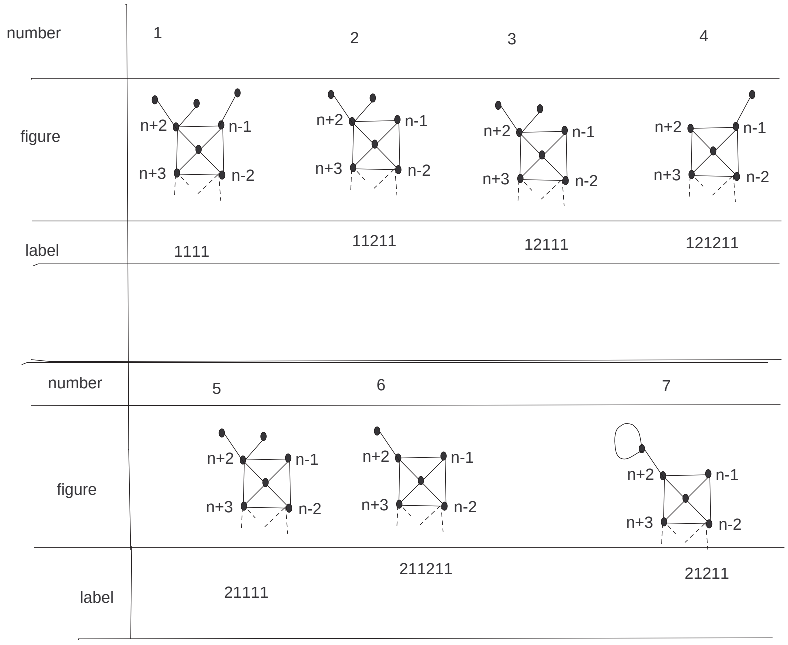


FIGURE 4. Table for T'_{n-1}

LEMMA 4.1.3. *Let $\Psi_2(x, y)$ be a two variable polynomial and Let $T(T'_{n-1}; x, y)$ be the Tutte polynomial of a triangulated $(n-1)$-ladder which the last rung is replaced by two parallel edges. Then*

$$\Psi_2(x, y) = [x^2 + 4x + 2xy + 3y + y^2 + 2]T(T'_{n-1}; x, y).$$

PROOF. See the definition of $\Psi_2(x,y)$ and the next table analyzing the results in Figure 4 □

number of terms	with loop	with coloop	corresponding term
1	0	2	x^2
4	0	1	$4x$
2	1	1	$2xy$
3	1	0	$3y$
1	2	0	$1y^2$
2	0	0	2

Finally we gather the terms in $T(T\text{"}_{n-1};x,y)$ as shown in Figure 5.

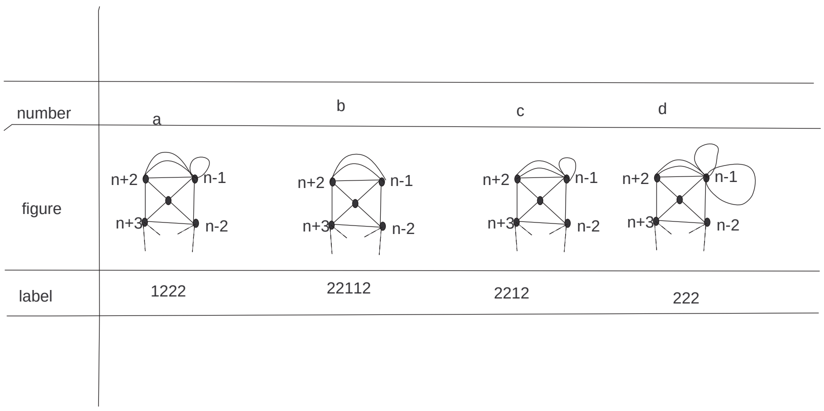


FIGURE 5. Table for $T\text{"}_{n-1}$

LEMMA 4.1.4. *Let $\Psi_3(x,y)$ be a two variable polynomial and Let $T(T\text{"}_{n-1};x,y)$ be the Tutte polynomial of a triangulated $(n-1)$-ladder which the last rung is replaced by 3 parallel edges. Then*

$$\Psi_3(x,y) = [y^2+2y+1]T(T\text{"}_{n-1};x,y).$$

PROOF. See the definition of $\Psi_3(x, y)$ and the following table analyzing the results in Figure 5 $\square$

number of terms	with loop	with coloop	corresponding term
1	2	0	y^2
2	1	0	$2y$
0	0	0	1

Now we are able to state and prove the main result of this chapter. Recall that T'_n and $T"_n$ are obtained from T_n by replacing the last rung by two and three parallel edges respectively.

PROPOSITION 4.1.5. *Let T_n be a triangulated n-ladder. Then the Tutte polynomial*

$$\begin{aligned} T(T_n; x, y) &= [x^3 + 3x^2 + xy + 2x]T(T_{n-1}; x, y) \\ &+ [x^2 + 4x + 2xy + 3y + y^2 + 2]T(T'_{n-1}; x, y) \\ &+ [y^2 + 2y + 1]T(T"_{n-1}; x, y). \end{aligned}$$

PROOF. We have seen that

$$T(T_n; x, y) = \Psi_1(x, y) + \Psi_2(x, y) + \Psi_3(x, y).$$

Therefore by applying Lemma 4.1.2, Lemma 4.1.3 and Lemma 4.1.4 we get

$$\begin{aligned} T(T_n; x, y) &= [x^3 + 3x^2 + xy + 2x]T(T_{n-1}; x, y) \\ &+ [x^2 + 4x + 2xy + 3y + y^2 + 2]T(T'_{n-1}; x, y) \\ &+ [y^2 + 2y + 1]T(T"_{n-1}; x, y) \end{aligned}$$

$\square$

4.1.2. Examples of Tutte polynomials. In this subsection we compute the Tutte polynomial of a triangulated 2-ladder using deletion and contraction formula. Then we calculate the same polynomial using Proposition 4.1.5 and show that the two are equal verifying our result. Here, we state the next lemma without proof, for further details, see Brylswski (1972).

LEMMA 4.1.6. *Let G be a coloop. Then*

$$T(G; x, y) = x.$$

LEMMA 4.1.7. *Let G be a graph on two vertices with two parallel edges only. Then*

$$T(G; x, y) = x + y.$$

PROOF. We refer to Figure 6 for the computation. Hence by ap-

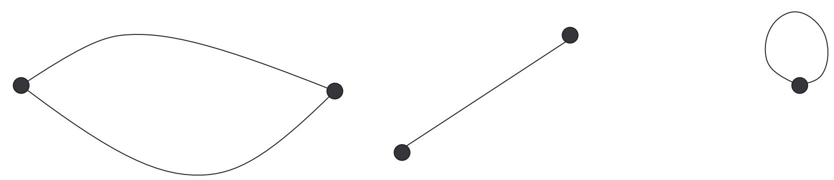


FIGURE 6

plying Proposition 2.2.2 and Lemma 4.1.6 we get

$$T(G; x, y) = x + y$$

□

LEMMA 4.1.8. *Let G be a graph on two vertices with three parallel edges only. Then*

$$T(G; x, y) = x + y + y^2.$$

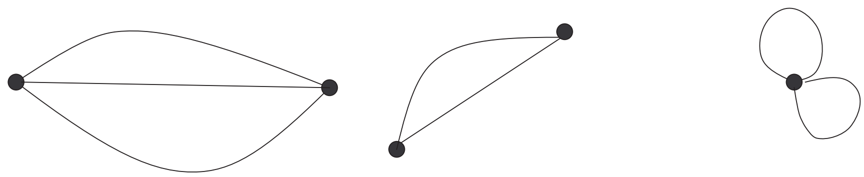


FIGURE 7

PROOF. We refer to Figure 7 for the computation. Hence by applying Proposition 2.2.2 and Lemma 4.1.7 we get

$$T(G;x,y) \quad = \quad x+y+y^2$$

$\square$

LEMMA 4.1.9. *Let H_1 be a 3-cycle. Then*

$$T(H_1;x,y) = x^2+x+y.$$

PROOF. We use the deletion-contraction formula for Tutte polynomial as shown in Figure 8. Hence by applying Proposition 2.2.2 and

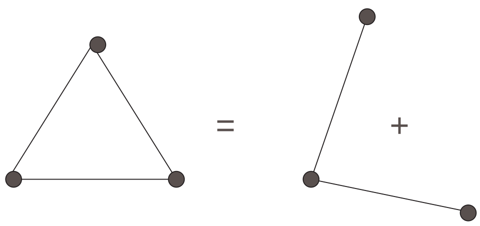


FIGURE 8

Lemma 4.1.7 we get

$$T(H_1; x, y) \quad = \quad x^2 + x + y$$

$\square$

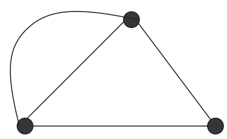


FIGURE 9. H_2

LEMMA 4.1.10. *Let H_2 be the graph shown in Figure 9. Then*

$$T(H_2; x, y) = [x^2 + x + y] + y(x + y).$$

PROOF. We use the deletion-contraction formula for the Tutte polynomial as shown in Figure 10. Hence by applying Proposition 2.2.2,

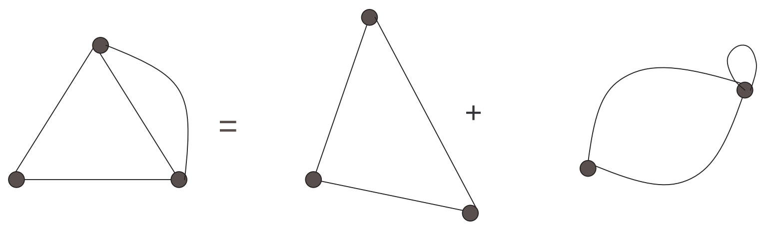


FIGURE 10

Lemma 4.1.7 and Lemma 4.1.9 we get

$$T(H_2; x, y) = [x^2 + x + y] + y(x + y)$$

$\square$

LEMMA 4.1.11. *Let H_3 be the graph shown in Figure 11. Then*

$$T(H_3; x, y) = (x^2 + x + y) + y(x + y) + y(x + y + y^2).$$

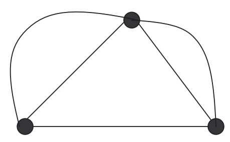


FIGURE 11. H_3

PROOF. We use the deletion-contraction formula for the Tutte polynomial as shown in Figure 12. Hence by applying Proposition 2.2.2,

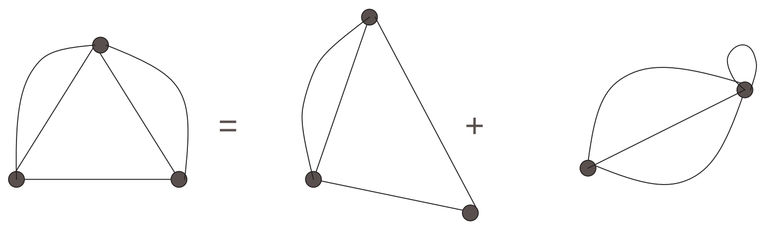


FIGURE 12

Lemma 4.1.8 and Lemma 4.1.10 we get

$$T(H_3; x, y) = (x^2 + x + y) + y(x + y) + y(x + y + y^2)$$

$\square$

LEMMA 4.1.12. *Let H_4 be the graph shown in Figure 13. Then*

$$T(H_4; x, y) = (x^2 + x + y) + y(x + y) + (y + y^2)(x + y + y^2).$$

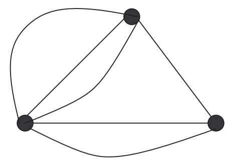


FIGURE 13. H_4

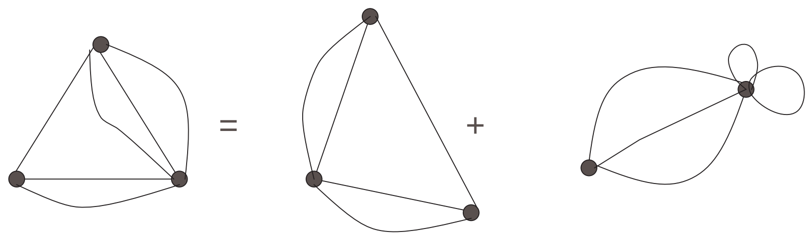


FIGURE 14

PROOF. We use the deletion-contraction formula for the Tutte polynomial as shown in Figure 14. Hence by applying Proposition 2.2.2, Lemma 4.1.8 and Lemma 4.1.11 we get

$$T(H_4; x, y) = (x^2 + x + y) + y(x + y) + (y + y^2)(x + y + y^2)$$

$\square$

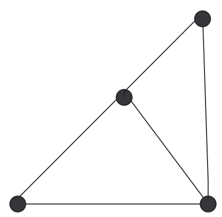


FIGURE 15. H_5

LEMMA 4.1.13. *Let H_5 be the graph shown in Figure 15. Then*

$$T(H_5;x,y) = (x+1)(x^2+x+y) + y(x+y).$$

PROOF. We use the deletion-contraction formula for the Tutte polynomial as shown in Figure 16. Hence by applying Proposition 2.2.2,

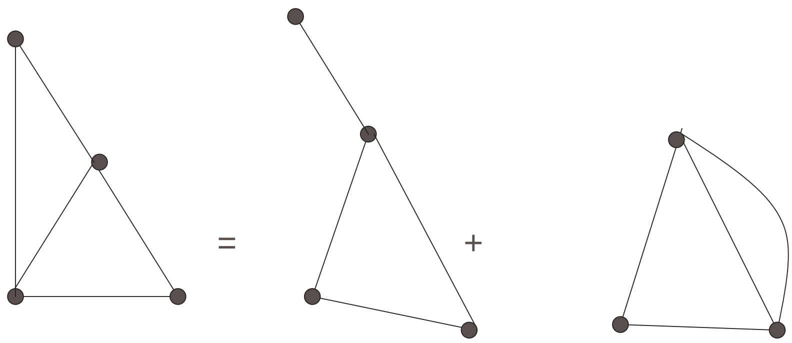


FIGURE 16

Lemma 4.1.9 and Lemma 4.1.10 we get

$$T(H_5;x,y) = (x+1)(x^2+x+y) + y(x+y)$$

□

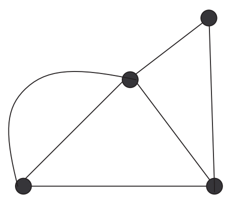


FIGURE 17. H_6

LEMMA 4.1.14. *Let H_6 be the graph shown in Figure 17. Then*

$$T(H_6;x,y) = (1+x+y)(x^2+x+y) + (y+y^2)(x+y).$$

PROOF. We use the deletion-contraction formula for the Tutte polynomial as shown in Figure 18. Hence by applying Proposition 2.2.2,

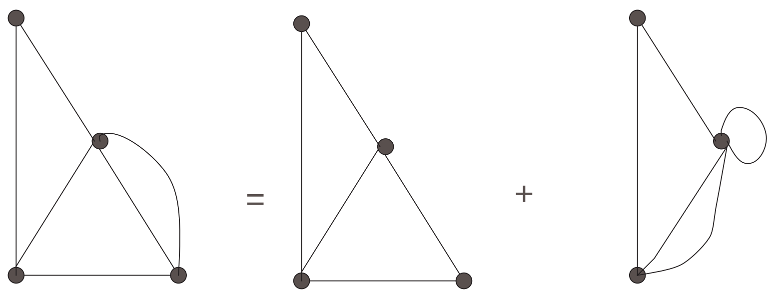


FIGURE 18

Lemma 4.1.10 and Lemma 4.1.13 we get

$$\begin{aligned} T(H_6; x, y) &= [(x+1)(x^2+x+y)+y(x+y)] \\ &+ y[(x^2+x+y)+y(x+y)] \\ &= (1+x+y)(x^2+x+y)+(y+y^2)(x+y) \end{aligned}$$

$\square$

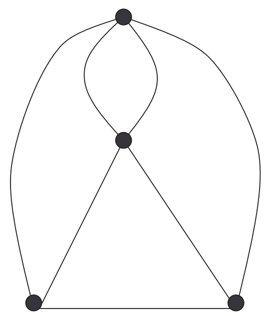


FIGURE 19. H_9

LEMMA 4.1.15. *Let H_9 be the graph shown in Figure 19. Then*

$$T(H_9; x, y) = (x+y+2)(x^2+x+y)+(2y+y^2)(x+y)+(y+y^2)(x+y+y^2).$$

PROOF. We use the deletion-contraction formula for the Tutte polynomial as shown in Figure 20. Hence by applying Proposition 2.2.2,

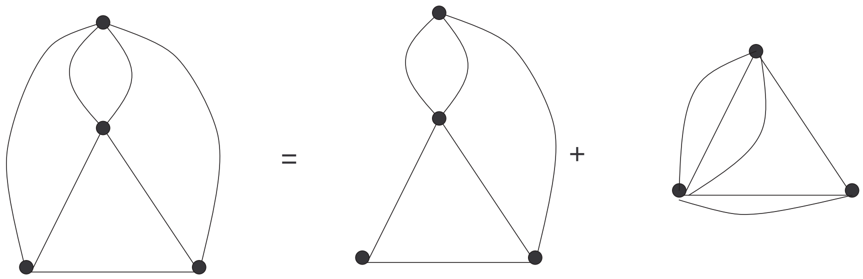


FIGURE 20

Lemma 4.1.14 and Lemma 4.1.12 we get

$$\begin{aligned}
T(H_9; x, y) &= [(1+x+y)(x^2+x+y)+(y+y^2)(x+y)] \\
&+ [(x^2+x+y)+y(x+y)+(y+y^2)(x+y+y^2)] \\
&= (x+y+2)(x^2+x+y)+(2y+y^2)(x+y) \\
&+ (y+y^2)(x+y+y^2)
\end{aligned}$$

$\square$

LEMMA 4.1.16. *Let H_{10} be the graph shown in Figure 21. Then*

$$T(H_{10}; x, y) = (x^2+2x+y+1)(x^2+x+y)+(xy+y+y^2)(x+y).$$

PROOF. We use the deletion-contraction formula for the Tutte polynomial as shown in Figure 22. Hence by applying Proposition 2.2.2,

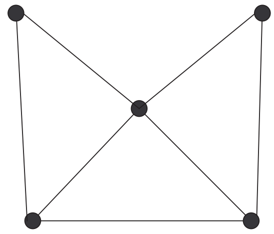
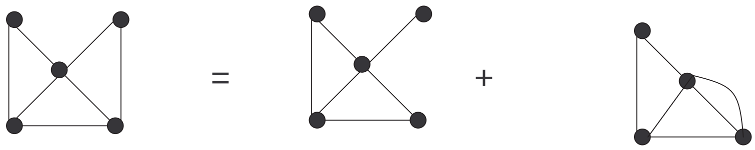


FIGURE 21. H_{10}



.

FIGURE 22

Lemma 4.1.13 and Lemma 4.1.14 we get

$$\begin{aligned} T(H_{10}; x, y) &= x[(x+1)(x^2+x+y) + xy(x+y)] \\ &+ [(1+x+y)(x^2+x+y) + (y+y^2)(x+y)] \\ &= (x^2+2x+y+1)(x^2+x+y) + (xy+y+y^2)(x+y) \end{aligned}$$

$\square$

Now we are able to give the Tutte polynomial for T_2 using deletion-contraction formula. Then we compare it with the Tutte polynomial for T_2 using Proposition 4.1.5.

EXAMPLE 4.1.17. Let T_2 be the triangulated 2-ladder. We use the deletion-contraction formula for the Tutte polynomial as shown in Figure 23. We observe that

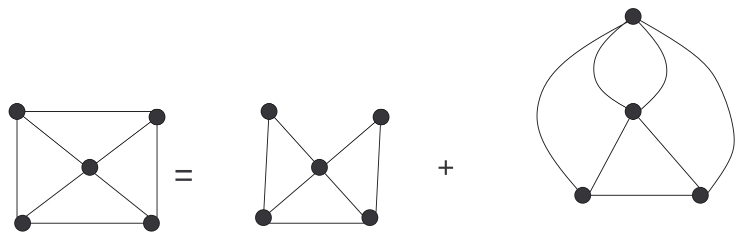


FIGURE 23

$$T(T_2; x, y) = T(H_{10}; x, y) + T(H_9; x, y).$$

But we have $T(H_{10}; x, y)$ and $T(H_9; x, y)$ by Lemma 4.1.16 and Lemma 4.1.15 respectively. Hence we substitute to get

$$\begin{aligned}
T(T_2; x, y) &= T(H_{10}; x, y) + T(H_9; x, y) \\
&= (x^2 + 2x + y + 1)(x^2 + x + y) \\
&+ (xy + y + y^2)(x + y)] \\
&+ (x + y + 2)(x^2 + x + y) + (2y + y^2)(x + y) \\
&+ (y + y^2)(x + y + y^2)] \\
&= (x^3 + 3x^2 + xy + 2x)x \\
&+ (x^2 + 4x + 2xy + 3y + y^2 + 2)(x + y) \\
&+ (y^2 + 2y + 1)(x + y + y^2).
\end{aligned}$$

Let T_2 be the triangulated 2-ladder. We use the Proposition 4.1.5 to compute the Tutte polynomial of T_2. We get

$$\begin{aligned} T(T_n;x,y) &= [x^3+3x^2+xy+2x]T(T_{n-1};x,y) \\ &+ [x^2+4x+2xy+3y+y^2+2]T(T'_{n-1};x,y) \\ &+ [y^2+2y+1]T(T''_{n-1};x,y). \end{aligned}$$

In this case T_1 is a coloop, T'_1 is a parallel pair of edges and $T"_1$ is a bunch of three parallel edges. The Tutte polynomials $T(T_1;x,y)$ is given by Lemma 4.1.6, $T(T'_1;x,y)$ is given by Lemma 4.1.7 and $T(T"_1;x,y)$ is given by Lemma 4.1.8. Hence

$$\begin{aligned} T(T_2;x,y) &= [x^3+3x^2+xy+2x]x \\ kauf &+ [x^2+4x+2xy+3y+y^2+2](x+y) \\ &+ [y^2+2y+1](x+y+y^2). \end{aligned}$$

Thus the Tutte polynomial $T(T_2;x,y)$ using deletion and contraction method is the same as the one found using Proposition 4.1.5.

4.2. Links associated with a triangulated ladder.

In this section we give the general notions of links which are relevant to this thesis. Then we give a brief outline on construction of links from planar graphs in general. Further, we construct a class of links corresponding to a triangulated n-ladder. Finally we study this class of links and give its component number. Notions defined in Section 4.2.1 and Section 4.2.2 are well known in knot theory. We refer the reader to Adams (1994) for further details.

4.2.1. General definitions on links. A *link* is defined as an embedding of n circles in $\mathbb{R}^3$. Each circle form a component of the link. A *knot* is a link with one such circle. We shall use the notation L for links and K for knots if we would like to distinguish a knot and a link. Otherwise we just use K when referring to either a link or a knot. The usual presentation of links involves a projection from the embedding space $\mathbb{R}^3$ to $\mathbb{R}^2$ with resultant apparent crossings of the curves on one or more embedded circles $S^{1's}$. This presentation in $\mathbb{R}^2$ is called a *link diagram* of our link. The simplest link is a link with one component and no crossing and is called *unknot* or *trivial knot*. It is apparent that when the curves crosses one component is on top of the other at that crossing. At any crossing of a link, the part of the curve which goes over the other is called an *overpass* and the one which goes under is called an *underpass*. In a link diagram, at any crossing, an overpass is indicated by a continuous line and an underpass is indicated by a broken line. A *link universe* is a diagram of link without any information on which string goes under or over. An *alternating link* is defined as one on which, as one travels along each of n embedded circles $S^{1's}$, one traverses alternatively between overpass and underpass. An *orientation* is defined by choosing a direction to travel around a link. This direction defined by placing coherently directed arrows along the link diagram in the direction of our choice. We say that a link is *oriented* if each component has an orientation. It is obvious that a link which is not oriented is called *unoriented link*.

EXAMPLE 4.2.1. The diagram in Figure 24 is an example of unoriented alternating link. At the labelled crossing C, the part of a curve labelled (d, e) is an over pass and the part labelled (a, b) is an underpass. The diagram in Figure 25 is an example of an oriented link. The diagram in Figure 26 is an example of link universe.

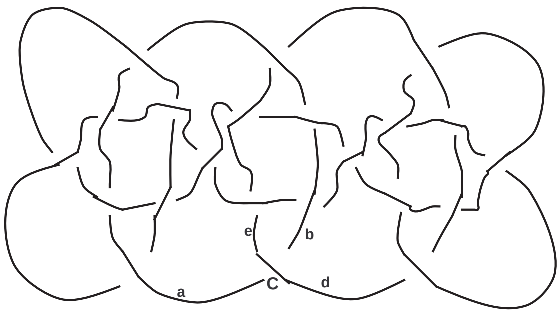


FIGURE 24

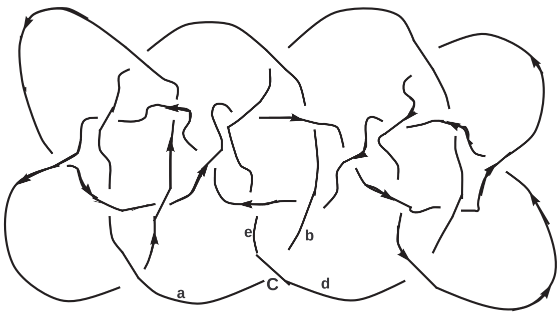


FIGURE 25

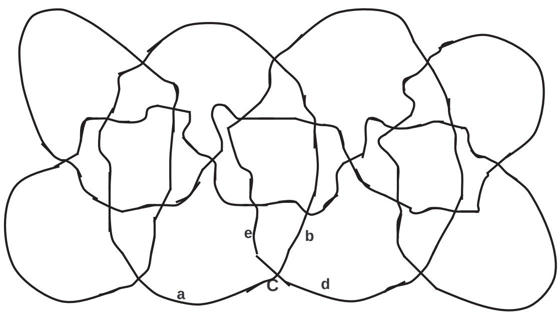


FIGURE 26

4.2.2. Construction of links from planar graphs. In this subsection we demonstrate the construction of a link diagram from a planar graph . Finally, we give an example of this construction by constructing links corresponding to a triangulated n-ladder. Let G a planar graph. Put an X across the midpoint of each edge of G. Now connect the end points of the X's in each region of the graph in such a way that neighbouring end points are connected. The result is a link universe. Then we can decide on which string to go under or over to create a diagram. The link obtained from G using this construction is denoted by $K(G)$. To demonstrate this construction we construct step by step a link corresponding to T_4

EXAMPLE 4.2.2. **Step 1.** We start with the planar graph T_4 as shown in Figure 27. **Step 2.** We put an X across each edge of T_4 as

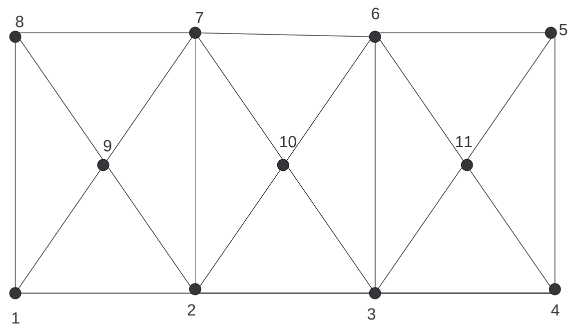


FIGURE 27

shown in Figure 28. **Step 3.** Now connect the end points of the X's in each region of T_4 in such a way that neighbouring end points are connected as shown in Figure 29. **Step 4.** Remove the graph T_4 to remain with a link universe $U(K(T_4))$ as shown in Figure 30. **Step 5.** Make the components to go over/under to create an alternating link $K(T_4)$ as shown in Figure 31

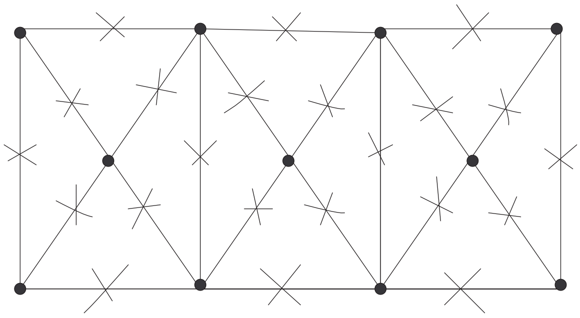


FIGURE 28

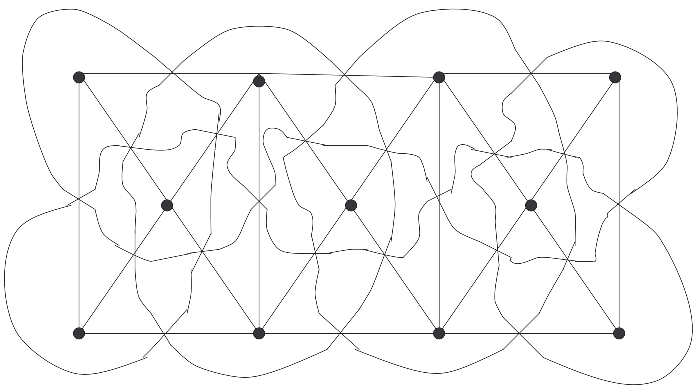


FIGURE 29

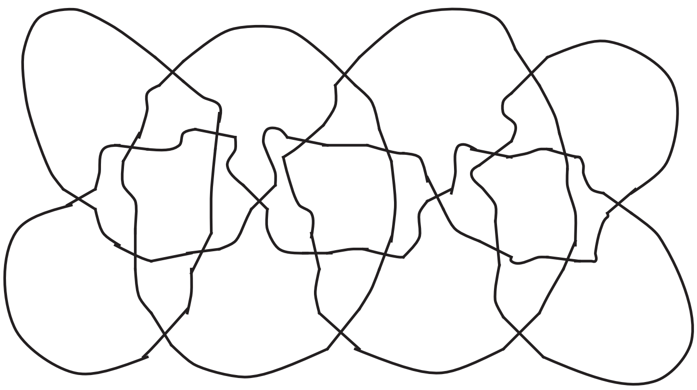


FIGURE 30

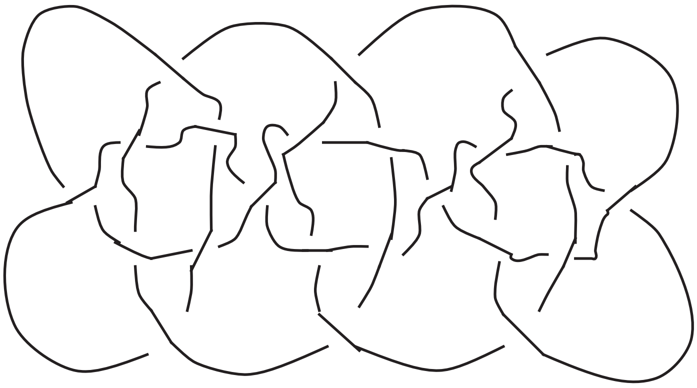


FIGURE 31

This construction is reversible, that is we can construct a planar graph from a link diagram. For futher details on this construction, we refer the reader to Chang and Shrok (2001).

4.2.3. Component number of links. In this subsection we give the component number of the link corresponding to a triangulated n-ladder. For any unoriented link diagram $K(G)$, a number

$$\mathcal{L}(K(G)) = (-1)^{|E(G)|}(-2)^{c-1}$$

can be defined, where c is the component number of $K(G)$ and $E(G)$ is the edge set of G, we are refer the reader to Mphako (2002) for more details. Thus by applying this to unoriented link diagram corresponding to a triangulated n-ladder T_n, we get

$$\mathcal{L}(K(T_n)) = (-1)^{|E(T_n)|}(-2)^{c_n-1}$$

where c_n is the component number of $K(T_n)$ and $E(T_n)$ the edge set of T_n. We need to state and prove some lemmas before stating and proving the one of the main results of this chapter. To ease notation, in this section, we shall denote, T_n' a graph obtained by adding an edge f parallel to the last rung $n, n+1)$ of T_n.

LEMMA 4.2.3. *Let $K(T_n)$ be a link diagram corresponding to T_n and let $K(T_n')$ be a link diagram associated to T_n'. Then,*

$$\mathcal{L}(K(T_n')) = -\mathcal{L}(K(T_n).$$

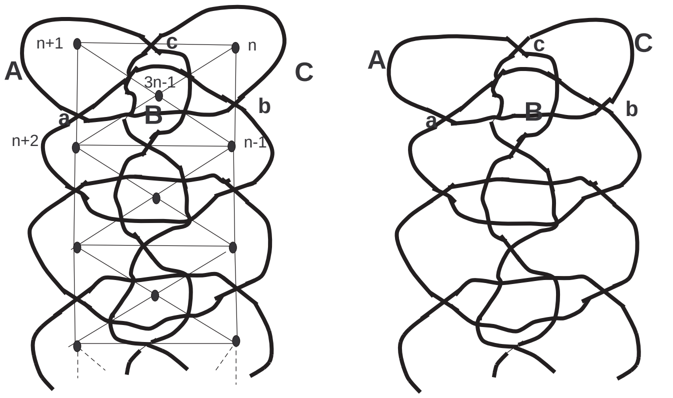


FIGURE 32. $K(T_n)$

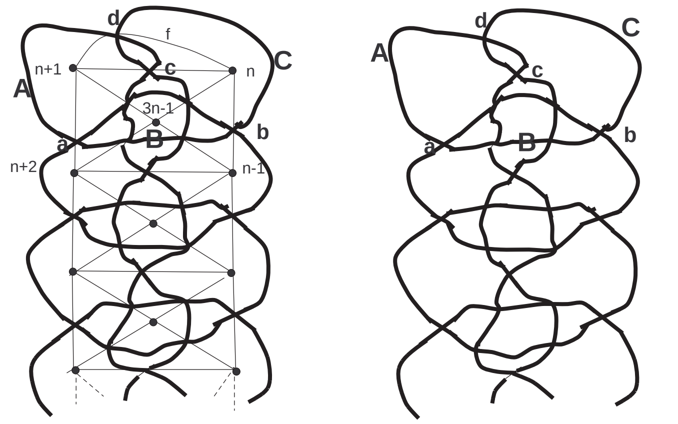


FIGURE 33. $K(T_n')$

PROOF. We use a link diagram $K(T_n)$ as shown in Figure 32 and link diagram $K(T_n')$ as shown in Figure 33. Recall that T_n' is just T_n

with a parallel edge on the last rung. We give both $K(T_n)$ and $K(T'_n)$ a similar labeling except at the crossing on the edge f which is only in T'_n. Let a, b, c, d be the crossing points as shown in the diagrams. String B run through from crossing point a to b. String A run through from crossing b to c and string C run through from crossing a to c, see diagrams. We observe that $A \cup B \cup C$ is one string according to our labeling in Figure 32. Using the same labeling in Figure 33 the new crossing d affects strings A and C by interchanging them. Thus the whole string $A \cup B \cup C$ remains unchanged. Hence the component number of $K(T_n)$ and $K(T'_n)$ is the same. Thus $c'_n = c_n$ where c'_n and c_n are component numbers of $K(T'_n)$ and $K(T_n)$, respectively. We know that $|E(T'_n)| = |E(T_n)| + 1$ by definition of T'_n. Hence

$$\begin{aligned}
\mathcal{L}(K(T'_n)) &= (-1)^{|E(T'n)|}(-2)^{c'_n-1} \\
&= (-1)^{|E(T_n)|+1}(-2)^{c_n-1} \\
&= (-1)(-1)^{E(T_n)}(-2)^{c_n-1} \\
&= (-1)\mathcal{L}(K(T_n))
\end{aligned}$$

□

The next two propositions demonstrate how to compute the number $\mathcal{L}(K(G))$ for a graph G. For further details see Mphako (2002). We need these propositions to prove the next lemma.

PROPOSITION 4.2.4. *If G is a planar graph, then*

(1) $\mathcal{L}(K(G)) = (-2)^{n-1}$ *if G is a graph with n vertices and no edge, thus $\mathcal{L}(K(G)) = 1$ if G is a single vertex.*
(2) $\mathcal{L}(K(G)) = -1$ *if G is an isthmus or a loop.*
(3) $\mathcal{L}(K(G)) = -\mathcal{L}(G\backslash e))$ *if e is a loop.*
(4) $\mathcal{L}(K(G)) = -\mathcal{L}(K(G/e))$ *if e is an isthmus and*

(5) $\mathcal{L}(K(G)) = \mathcal{L}(K(G\backslash e)) + \mathcal{L}(K(G/e))$ *if e is neither a loop nor an isthmus.*

PROPOSITION 4.2.5. *Let G be a planar graph. Then*

(1) $\mathcal{L}(K(G)) = \mathcal{L}(K(G/e/f))$ *if e and f are series pair; and*

(2) $\mathcal{L}(K(G)) = \mathcal{L}(K(G\backslash e\backslash f))$ *if e and f are parallel pair.*

LEMMA 4.2.6. *Let $K(T_i)$ be a link diagram corresponding to T_i, where $\{i \in \mathbb{Z} | i > 1\}$. Then $c_n = c_{n-1} + 1$ where c_i is the number of components of the link $K(T_i)$.*

PROOF. To prove this proposition, we apply Equation 5 of Proposition 4.2.4. Let T_n be a triangulated n-ladder and let e be an end bar edge, then

$$\mathcal{L}(K(T_n)) = \mathcal{L}(K(T_n\backslash e)) + \mathcal{L}(K(T_n/e)).$$

Recall that a diagram of a graph G represent the number $\mathcal{L}(K(G))$. We have an equation as shown in Figure 34. Now we compute $\mathcal{L}(K(T_n\backslash e))$

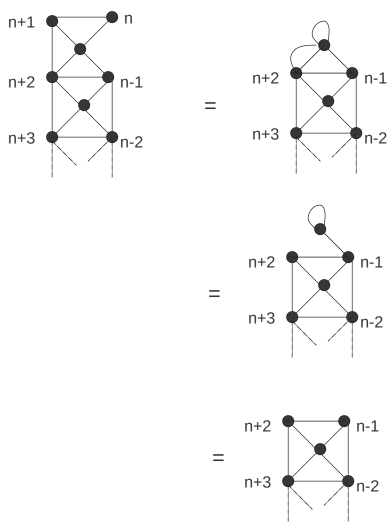


FIGURE 34

as shown in Figure 34 and applying Proposition 4.2.5 and Proposition 4.2.4. We start by contracting a series pair. Then we delete a parallel pair. Finally we contract an isthmus and delete a loop. Hence, as shown by the Figure 34

$$\mathcal{L}(K(T_n\backslash e)) = \mathcal{L}(K(T_{n-1})).$$

Next step, we calculate $\mathcal{L}(K(T_n/e))$ as shown in Figure 35 and applying Proposition 4.2.5 and Proposition 4.2.4. We start by deletion of a parallel pair. Then we contract a series pair. Finally we delete of a loop. Hence, as shown by the Figure 35

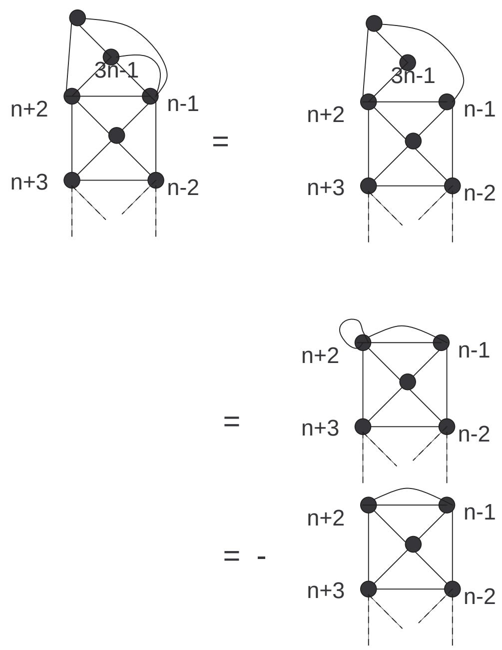


FIGURE 35

$$\mathcal{L}(K(T_n/e)) = -\mathcal{L}(K(T'_{n-1})).$$

But by applying Lemma 4.2.3, we get

$$\begin{aligned}
\mathcal{L}(K(T_n/e)) &= -\mathcal{L}(K(T'_{n-1})) \\
&= -(-\mathcal{L}(K(T_{n-1}))) \\
&= \mathcal{L}(K(T_{n-1}))).
\end{aligned}$$

Now recall that c_n denote the number of components of the link $K(T_n)$ and substituting in the equation

$$\mathcal{L}(K(T_n)) = \mathcal{L}(K(T_n \backslash e)) + \mathcal{L}(K(T_n/e)).$$

We get

$$\begin{aligned}
\mathcal{L}(K(T_n)) &= \mathcal{L}(K(T_{n-1})) - \mathcal{L}(K(T'_{n-1})) \\
&= \mathcal{L}(K(T_{n-1})) + \mathcal{L}(K(T_{n-1})) \\
&= 2\mathcal{L}(K(T_{n-1})) \\
&= 2(-1)^{|E(T_{n-1})|}(-2)^{c_{n-1}-1}.
\end{aligned}$$

By Proposition 3.1.4, we know that $|E(T_n)| = |E(T_{n-1})| + 7$. Thus $|E(T_{n-1})| = |E(T_n)| - 7$. It follows directly that

$$(-1)^{|E(T_{n-1})|} = (-1)^{|E(T_n)|-7} = (-1)^{|E(T_n)|-1}.$$

Therefore

$$\begin{aligned}
\mathcal{L}(K(T_n)) &= 2(-1)^{|E(T_n)|-1}(-2)^{(c_{n-1})-1} \\
&= -2(-1)^{|E(T_n)|}(-2)^{(c_{n-1})-1} \\
&= (-1)^{|E(T_n)|}(-2)^{[(c_{n-1})+1]-1}.
\end{aligned}$$

But by definition

$$\mathcal{L}(K(T_n)) = (-1)^{|E(T_n)|}(-2)^{(c_n-1)}.$$

Thus

$$c_{n-1} + 1 = c_n.$$

and

$$c_n - 1 = c_{n-1}$$

□

The next lemma is stated without proof, for further details and proof, see [**?**].

PROPOSITION 4.2.7. *Let G be an n-wheel. Then $K(G)$ is a 3-link if n is divisible by 3. Otherwise it is a knot.*

PROPOSITION 4.2.8. *Let $K(T_n)$ be a link diagram corresponding to a triangulated n-ladder. Then, for $n > 1$,*

$$c_n = n - 1$$

where c_n is the number of components of the link $K(T_n)$.

PROOF. The proof is by induction on the number n. Let $n = 2$. Then T_2 is a 4-wheel. By applying Proposition 4.2.7 a link corresponding to a 4-wheel is a knot. Therefore it has 1 component by definition of a knot. Thus $c_2 = 1 = 2 - 1$. Hence proposition is true for the base case. Assume that the proposition is true for some $n = k$. That is $c_k = k - 1$. Now consider $n = k + 1$. Let $K(T_{k+1})$ be a link diagram of T_{k+1} and c_{k+1} its component number. By Lemma 4.2.6 $c_{k+1} = c_k + 1$. Hence by induction hypothesis $c_{k+1} = (k - 1) + 1 = k$. Therefore the proposition is true for any $n > 1$ □

CHAPTER 5

Conclusion.

We found a factor $(\lambda - 2)(\lambda^2 - 5\lambda + 7)$ to be multiplied by the chromatic polynomial of a triangulated ladder of one size less, to get the recursive expression of a given triangulated ladder, and the same factor to the power $n - 1$, to be multiplied by λ and get the explicit expression of a given triangulated ladder. The Tutte and the coboundary polynomials are polynomials on two variables, and are expressed respectively, as a sum of three Tutte and coboundary polynomials of triangulated ladders of one size less, up to the parallel classes for the last rung, with suitable factors as polynomials of the same variables. The component number of a diagram link associated to a triangulated ladder is the one less the size of that triangulated ladder.

This work is not exhaustive; we would like to go further in establishing the recursive expressions of various polynomials we studied. Indeed, the recursive expressions of these polynomials are presented on the form of a sum of three different polynomials. Further research is needed to find out the relationship between parallel class of triangulated ladders we mentioned, in order to get a formal and unified recursive expression of our polynomials. Among various graphs, Triangulated graphs have a big range of applications in Science and Technology; therefore many topics of them have to be explored.

Index of Definitions

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